Approximating a continuously stratified hydrostatic system by the multi-layer shallow water system

Mahieddine Adim
Under the supervision of
Vincent Duchêne

4 Octobre 2023

Université de Rennes
Goal: Establishing a bridge between multilayer and continuously stratified hydrostatic flows.

Considered systems:

- The continuously stratified hydrostatic system (CSHS).
- The multi-layer shallow water system (MSWS).
\[(CSHS+GM)\]

\[
\begin{aligned}
\begin{cases}
\partial_t h + \partial_x ((h + h)(u + u)) = \kappa \partial_x^2 h \\
\partial_t u + \left(u + u - \kappa \frac{\partial_x h}{h + h}\right) \partial_x u + \frac{1}{\bar{\rho}} \mathcal{M}(\partial_x h) = 0
\end{cases}
\end{aligned}
\]

Where

\[
\mathcal{M}(\partial_x h(t, x))(\varrho) = \int_{\rho_{surf}}^{\rho_{bott}} \min(\rho, \rho') \partial_x h(t, x, \varrho') d\varrho'.
\]

\[(MSWS+GM)\]

\[
\begin{aligned}
\begin{cases}
\partial_t H_i + \partial_x ((H_i + H_i)(U_i + U_i)) = \kappa \partial_x^2 H_i, \quad \forall i \in \{1, \cdots, N\} \\
\partial_t U_i + \left(U_i + U_i - \kappa \frac{\partial_x H_i}{H_i + H_i}\right) \partial_x U_i + g \sum_{j=1}^{N} \frac{1}{N} \frac{\min(\rho_i, \rho_j)}{\rho_i} \partial_x H_j = 0.
\end{cases}
\end{aligned}
\]
\[
\begin{align*}
\left\{ \begin{array}{l}
\partial_t H_i + \partial_x ((H_i + H_i)(U_i + U_i)) = \kappa \partial_x^2 H_i, \quad \forall i \in \{1, \cdots, N\} \\
\partial_t U_i + \left( U_i + U_i - \kappa \frac{\partial_x H_i}{H_i + H_i} \right) \partial_x U_i + g \sum_{j=1}^{N} \frac{1}{N} \frac{\min(\rho_i, \rho_j)}{\rho_i} \partial_x H_j = 0.
\end{array} \right.
\end{align*}
\]

Where

- \( t > 0, \ x \in \mathbb{R} \).
- \( H_i, \ U_i, \ \rho_i \in \mathbb{R} \).
- \( H_i, \ U_i \) are the deviation of the equilibrium \( \underline{H}_i, \ \underline{U}_i \).
- \( g \) denotes the acceleration of gravity.
- The densities satisfy \( \rho_i = \rho_{\text{surf}} + \frac{i-1}{N} (\rho_{\text{bott}} - \rho_{\text{surf}}) \)
  \( \forall i \in \{1, \cdots, N\} \), with \( \rho_{\text{bott}} > \rho_{\text{surf}} > 0, \ \rho_{\text{bott}} - \rho_{\text{surf}} = 1 \).
Remarks

- The $\kappa$ terms are motivated by the work of the oceanographers Gent & McWilliams on isopycnal mixing and eddy diffusivity (in the 90’s), and which could be interpreted as turbulence terms. Moreover the adding of this diffusive $\kappa$ term in the first equation has a regularizing effect.

- The system $(\text{CSHS} + \text{GM})$ is well-posed in Sobolev spaces on the time interval $[0, T]$ with $T^{-1} = C(1 + \kappa^{-1}(|u'|_{L^2}^2 + M_0^2))$, where $M_0$ is the size of the initial data, and $C$ depends only on $M_0$ and the size of the equilibrium $(\rho, u)$. [Duchêne & Bianchini '22].
Remarks

- The $\kappa$ terms are motivated by the work of the oceanographers Gent & McWilliams on isopycnal mixing and eddy diffusivity (in the 90’s), and which could be interpreted as turbulence terms. Moreover the adding of this diffusive $\kappa$ term in the first equation has a regularizing effect.

- The system (CSHS+GM) is well-posed in Sobolev spaces on the time interval $[0, T]$ with $T^{-1} = C(1 + \kappa^{-1}(|u'|^2_{L^2} + M_0^2))$, where $M_0$ is the size of the initial data, and $C$ depends only on $M_0$ and the size of the equilibrium $(\underline{\rho}, \underline{u})$. [Duchêne & Bianchini ’22].
The continuous case

\[ \rho \]

\[ h \]

\[ u \]

\[ \rho_{\text{bott}} \]

\[ \int_{\rho} f \, d\rho' \]

\[ \partial_\rho f \]

\[ \mathcal{M} f(\rho) = \int_{\rho_{\text{surf}}} \rho_{\text{bott}} \min(\rho, \rho') f(\rho') \, d\rho' \]

The discrete case

\[ \rho = (\rho_1, \cdots, \rho_N)^t \]

\[ (H_i)_{i=1,\cdots,N} = (h(\rho_i))_{i=1,\cdots,N} \]

\[ (U_i)_{i=1,\cdots,N} = (u(\rho_i))_{i=1,\cdots,N} \]

\[ (SF)_i = \sum_{j=i}^{N} (\rho_j - \rho_{j-1}) \, F_{j-1} \]

\[ (D_\rho F)_i = \frac{1}{\rho_i - \rho_{i+1}} (F_i - F_{i+1}) \]

\[ \sum_{j=1}^{N} \frac{\min(\rho_i, \rho_j)}{N} F_j \]
Main result

\((\text{MSWS}+\text{GM})\)

\[
\begin{aligned}
\partial_t H_i + \partial_x ((H_i + H_i)(U_i + U_i)) &= \kappa \partial_x^2 H_i, \quad \forall i \in \{1, \ldots, N\} \\
\partial_t U_i + \left( U_i + U_i - \kappa \frac{\partial_x H_i}{H_i + H_i} \right) \partial_x U_i + g \sum_{j=1}^{N} \frac{1}{N} \min(\rho_i, \rho_j) \rho_i \partial_x H_j &= 0.
\end{aligned}
\]

[MA ’23]

Let \(s \in \mathbb{N}\) be such that \(s > 2 + \frac{1}{2}\), there exists \(C > 0\) such that for any \(N \in \mathbb{N}^*\) and any \(\kappa > 0\), for any initial data \((H_0, U_0) \in H^{s,2}\), satisfying natural assumptions and with \(M_0 := \left\| (H_0, U_0) \right\|_s\), the following holds. Denoting

\[
T^{-1} = C \left( 1 + \kappa^{-1} \left( \left\| D_\rho U \right\|_2^2 + M_0^2 \right) \right),
\]

there exists a unique strong solution \((H, U)\) to \((\text{MSWS}+\text{GM})\) with initial data \((H, U)|_{t=0} = (H_0, U_0)\). and one has, for any \(t \in [0, T]\) the estimate \(\left\| (H, U)(t, \cdot) \right\|_s \leq CM_0\).
Remarks

- We have

\[ |||(H, U)|||_s = \sum_{j=0}^{1} \| D^j \rho H \|_{H^{s-j}_x} + \sum_{j=0}^{2} \| D^j \rho S H \|_{H^{s-j}_x} \]

\[ + \sum_{j=0}^{2} \| D^j \rho U \|_{H^{s-j}_x} + \| TSH \|_{H^s_x}. \]

- The time of existence is independent of the number of layers \( N \).

- There is an obvious similarity between the time

\[ T^{-1} = C \left( 1 + \kappa^{-1} \left( \| D_{\rho} U \|_{L^2}^2 + M_0^2 \right) \right) \]

obtained in the previous Theorem and the one of the WP of \((CSHS+GM)\).
\[ \begin{align*}
\partial_t H_i + \partial_x ((H_i + H_i)(U_i + U_i)) &= \kappa \partial_x^2 H_i, \quad \forall i \in \{1, \cdots, N\} \\
\partial_t U_i + \left( U_i + U_i - \kappa \frac{\partial_x H_i}{H_i + H_i} \right) \partial_x U_i + g \sum_{j=1}^{N} \frac{1}{N} \frac{\min(\rho_i, \rho_j)}{\rho_i} \partial_x H_j &= 0.
\end{align*} \]

**Tools of the proof**

- The **small time** existence and uniqueness part of the proof relies on existence and uniqueness theorems of transport and transport diffusion equations with their corresponding estimates.

- The **long time** existence and uniqueness is based on the energy method. Using our correspondence (Dictionary) the estimates derive naturally from the the estimates found in the continuous case \[Duchêne & Bianchini '22\].
Question: What happens when the number of layers $N$ tends to infinity?
Let $s \in \mathbb{N}$ such that $s > 2 + \frac{1}{2}$, and $\kappa > 0$. Moreover, consider regular and controlled $(h, u)$ on $(\rho_{surf}, \rho_{bott})$, and $(h, u)$ a strong sufficiently regular solution to $(\text{CSHS}+\text{GM})$ on a time interval $[0, T]$ with $T > 0$, satisfying natural assumptions.

Then there exists $N_0(T, \kappa) \in \mathbb{N}^*$ such that for all $N \geq N_0$ and any initial data $(H_0, U_0) = (P_N(h_0), P_N(u_0)) \in H^s(\mathbb{R})^{2N}$ the solution to $(\text{MSWS}+\text{GM})$ with $H = P_N(h)$, $U = P_N(u)$ and satisfying $(H, U)_{t=0} = (H_0, U_0)$ defined in the previous Theorem is well-defined on the time interval $[0, T]$ and satisfies for any $t \in [0, T]$

$$\|\| (H - P_N h, U - P_N u) \|\|_s(t) = O_{\kappa, T} \left( \frac{1}{N^2} \right),$$

where $P_N : C[\rho_{surf}, \rho_{bott}] \to \mathbb{R}^N$

$$f \mapsto P_N(f) = (f(\rho_i))_{1 \leq i \leq N}.$$
Tools of the proof:

- **Consistency**
  - A careful and precise analysis is done to obtain the rate $\frac{1}{N^2}$.
  - The same result can be obtained for other choices of the operator $P_N$ for instance $(P_N(f))_i = \frac{1}{\rho_{i+1}-\rho_i} \int_{\rho_i}^{\rho_{i+1}} f(\rho) d\rho$ but we may lose the following property $P_N(fg) = P_N(f)P_N(g)$.

- **Stability estimates**
  - When we consider the system of the difference between the solutions we obtain the same structure of equations as in the continuous case.
  - The estimates of this difference derive naturally from the estimates found in the continuous case [Duchêne & Bianchini '22].
Conclusion and perspective:

We rigorously justified the \((\text{MSWS} + \text{GM})\) as an approximation of the \((\text{CSHS} + \text{GM})\) when the number of layers \(N\) tends to infinity.

The current perspective is rigorously justify the convergence of \((\text{CSHS} + \text{GM})\) to \((\text{MSWS} + \text{GM})\) when we consider a continuous density that converges to a piecewise continuous density.

Thank you for your attention