The hydrostatic approximation for stratified fluids

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Joint work with Roberta Bianchini (CNR, Rome)
Outline

1. Hydrostatic equations
2. Eddy diffusivity
3. Isopycnal coordinates
4. Results
Motivated by geophysical flows, we consider the heterogeneous incompressible Euler equations with gravity force

\[
\begin{align*}
\partial_t \rho + \partial_x (\rho u) + \partial_z (\rho w) &= 0, \\
\rho \left( \partial_t u + u \partial_x u + w \partial_z u \right) + \partial_x P &= 0, \\
\rho \left( \partial_t w + u \partial_x w + w \partial_z w \right) + \partial_z P + g \rho &= 0, \\
\partial_x u + \partial_z w &= 0,
\end{align*}
\]

(E)

boundary conditions.

The velocity field is denoted \((u, w) : \Omega \rightarrow \mathbb{R} \times \mathbb{R}\), the density \(\rho : \Omega \rightarrow \mathbb{R}^+_\star\).

Reconstructing the pressure \(P : \Omega \rightarrow \mathbb{R}\) requires to solve an elliptic problem. Using instead the hydrostatic approximation, the system becomes ...
The hydrostatic approximation

The heterogeneous, incompressible Euler equations with gravity force

\[
\begin{align*}
\partial_t \rho + \partial_x (\rho u) + \partial_z (\rho w) &= 0, \\
\rho (\partial_t u + u \partial_x u + w \partial_z u) + \partial_x P &= 0, \\
\rho (\partial_t w + u \partial_x w + w \partial_z w) + \partial_z P + g \rho &= 0, \\
\partial_x u + \partial_z w &= 0,
\end{align*}
\]

(E)

Using the hydrostatic approximation, the system becomes

\[
\begin{align*}
\partial_t \rho + \partial_x (\rho u) + \partial_z (\rho w) &= 0, \\
\rho (\partial_t u + u \partial_x u + w \partial_z u) + \partial_x P &= 0, \\
\partial_z P &= -g \rho, \quad \iff \quad P = P|_{z=z_{\text{surf}}} + g \int_{z_{\text{surf}}}^{z_{\text{surf}}} \rho \, dz \\
\partial_z w &= -\partial_x u, \quad \iff \quad w = w|_{z=z_{\text{bot}}} - \int_{z_{\text{bot}}}^{z_{\text{bot}}} \partial_x u \, dz
\end{align*}
\]

(H)

boundary conditions.

**Pb:** stability of shear flows \((\rho, u)(t, x, z) = (\rho(z), u(z))\)?
Some stability results

Homogeneous case: \( \rho \equiv 1 \).

- Spectral stability of the linearized system about shear flows, under the Rayleigh criterion \( u''(z) \neq 0 \). [Rayleigh (1880)]
- Lyapunov stability under the Rayleigh criterion. [Arnold '65]
- Well-posedness of the (nonlinear) system in Sobolev spaces under the Rayleigh criterion. [Grenier '99], [Brenier '03], [Masmoudi&Wong '12].

Inhomogeneous case: \( \partial_z \rho \neq 0 \).

- Spectral stability of the linearized system about shear flows, under the Miles and Howard criterion \( \frac{1}{4}|u'(z)|^2 \leq \frac{-\rho'(z)}{\rho(z)}. [Miles '61][Howard '61]

Open problem

Well-posedness of the initial-value problem for the (linear or nonlinear) system?
Possible source of instabilities

The hydrostatic limit is singular:

\[
\begin{align*}
\partial_t \rho + u \partial_x \rho + w \partial_z \rho &= 0, \\
\rho \left( \partial_t u + u \partial_x u + w \partial_z u \right) + \partial_x P &= 0, \\
\partial_z P &= -g \rho, \quad \iff \quad P = P|_{z=z_{\text{surf}}} + g \int_{z_{\text{surf}}}^{z} \rho \, dz \\
\partial_z w &= -\partial_x u, \quad \iff \quad w = w|_{z=z_{\text{bot}}} - \int_{z_{\text{bot}}}^{z} \partial_x u \, dz
\end{align*}
\]

boundary conditions.

- Contributions from \( w \partial_z \rho \) and \( \partial_x P \) compensate in energy estimates if \( \rho > 0 \), \( \partial_z \rho < 0 \). Stable stratification helps.

- There is no obvious way to deal with the contribution \( w \partial_z u \). Shear velocity hurts.

- In the literature, theoretical results either
  1. Use the analytic framework. e.g. [Kukavica, Temam, Vicol & Ziane ’11]
  2. Add regularization (viscosity, diffusivity). e.g. [Cao, Li & Titi ’07+...]
  3. Relax the hydrostatic approximation [Desjardins, Lannes & Saut ’21]
We add to the system the contributions of non-resolved eddies suggested by [Gent & McWilliams ’90]. They act as additional “bolus” velocities.

\[
\begin{align*}
(H_{\kappa}) & \\
\partial_t \rho + (u + u_*) \partial_x \rho + (w + w_*) \partial_z \rho &= 0, \\
\rho (\partial_t u + (u + u_*) \partial_x u + (w + w_*) \partial_z u) + \partial_x P &= 0, \\
\partial_z P &= -g \rho, \quad \partial_z w = -\partial_x u,
\end{align*}
\]

boundary conditions (free surface).

\[
u_* = \kappa \partial_z \left( \frac{\partial_x \rho}{\partial_z \rho} \right), \quad w_* = -\kappa \partial_x \left( \frac{\partial_x \rho}{\partial_z \rho} \right),
\]

where \( \kappa > 0 \).
Isopycnal coordinates

We consider **stratified flows** and define the variable $h(t, x, \varrho) > 0$ through

$$h \overset{\text{def}}{=} -\partial_\varrho \zeta, \quad \rho(t, x, \zeta(t, x, \varrho)) = \varrho, \quad \zeta(t, x, \rho(t, x, z)) = z.$$
Isopycnal coordinates

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\]

The system

\[
\begin{align*}
\partial_t \rho + (u + u_\star) \partial_x \rho + (w + w_\star) \partial_z \rho &= 0, \\
\rho \left( \partial_t u + (u + u_\star) \partial_x u + (w + w_\star) \partial_z u \right) + \partial_x P &= 0, \\
\partial_z P &= -g \rho, \quad \partial_z w = -\partial_x u,
\end{align*}
\]

(reads in isopycnal coordinates)

\[
\begin{align*}
\partial_t h + \partial_x \left( h(u + u_\star) \right) &= 0, \\
\partial_t u + (u + u_\star) \partial_x u + \partial_x (\mathcal{M} h) &= 0,
\end{align*}
\]

where \( u_\star \overset{\text{def}}{=} \kappa \frac{-\partial_x h}{h} \) and

\[
(\mathcal{M} h)(t, x, \rho) \overset{\text{def}}{=} g \int_{\rho_0}^{\rho_1} \frac{\min(\rho, \rho')}{\rho} h(t, x, \rho') \, d\rho'
\]
The system in isopycnal coordinates

The isopycnal formulation of the hydrostatic system is

\[
(H_\kappa) \quad \begin{cases} 
\partial_t h + u \partial_x h + h \partial_x u = \kappa \partial^2_x h, \\
\partial_t u + (u + \kappa \frac{-\partial_x h}{h}) \partial_x u + \partial_x (\mathcal{M} h) = 0,
\end{cases}
\]

where \( \mathcal{M} h(t, x, \varrho) \overset{\text{def}}{=} g \int_{\varrho_0}^{\varrho_1} \min(\varrho, \varrho') \frac{h(t, x, \varrho')}{\varrho} \, d\varrho'. \)

Remarks:

- The parameterization of [Gent & McWilliams '90] is nice and simple.
- The advection in the variable \( z \) (or \( \varrho \)) has disappeared.
- The domain is flattened.
- The system is easily discretized (multilayer framework) [Adim]
- The structure of the system is clarified:

\[
\partial_t \begin{pmatrix} h \\ u \end{pmatrix} + \begin{pmatrix} 0 & \text{Diag}(h) \\ \mathcal{M} & 0 \end{pmatrix} \partial_x \begin{pmatrix} h \\ u \end{pmatrix} + \begin{pmatrix} \text{Diag}(u) & 0 \\ 0 & \text{Diag}(u) \end{pmatrix} \partial_x \begin{pmatrix} h \\ u \end{pmatrix} = \kappa \begin{pmatrix} \partial^2_x h \\ \frac{-\partial_x h}{h} \end{pmatrix}
\]
The isopycnal formulation of the hydrostatic system is

\[ \partial_t \begin{pmatrix} h \\ u \end{pmatrix} + \begin{pmatrix} 0 & \text{Diag}(h) \\ \mathcal{M} & 0 \end{pmatrix} \partial_x \begin{pmatrix} h \\ u \end{pmatrix} + \begin{pmatrix} \text{Diag}(u) & 0 \\ 0 & \text{Diag}(u) \end{pmatrix} \partial_x \begin{pmatrix} h \\ u \end{pmatrix} = \kappa \begin{pmatrix} \frac{\partial^2 h}{\partial_x h} \\ \frac{\partial h}{\partial_x h} \end{pmatrix} \]

1. The first two terms generate wave propagation

2. Advection contribution mixes modes when \( \partial_\rho u \neq 0 \).

3. The Gent&McWilliams contribution provides (partial) diffusion.\(^1\)

\(^1\)On a side note, if we define \( \nu = u - \kappa \frac{\partial_x h}{h} \), then the system becomes

\[ \partial_t \begin{pmatrix} h \\ \nu \end{pmatrix} + \begin{pmatrix} 0 & \text{Diag}(h) \\ \mathcal{M} & 0 \end{pmatrix} \partial_x \begin{pmatrix} h \\ \nu \end{pmatrix} + \begin{pmatrix} \text{Diag}(\nu) & 0 \\ 0 & \text{Diag}(\nu - \kappa \frac{\partial_x h}{h}) \end{pmatrix} \partial_x \begin{pmatrix} h \\ \nu \end{pmatrix} = \kappa \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \]

and the Gent&McWilliams contribution acts as an effective viscosity on the variable \( \nu \).
Some results

For sufficiently regular data satisfying the stable stratification assumption

\[ h|_{t=0} \geq h_0 > 0 \]

and for any \( \kappa > 0 \), there exists a unique (classical) solution which we control on the time interval \([0, T]\) with

\[ T^{-1} = C \left( 1 + \kappa^{-1} \left( |u'|^2_{L^2} + M_0^2 \right) \right), \]

where \( M_0 \) is the size of the initial deviation from the shear flow equilibrium \((\rho(\varrho), u(\varrho))\), and \( C \) depends only on \( M_0, h_0 \) and the size of \((\rho(\varrho), u(\varrho))\).

As long as this solution is controlled, we have strong convergence of the corresponding solutions to the incompressible Euler equations towards this solution in the limit of shallow water aspect ratio \( L_z/L_x \ll 1 \).
Some results

[R. Bianchini & VD]

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Thoughts to go

- We do not understand well the stability of the hydrostatic equations. Stable stratification helps. Shear velocity hurts.
- The Gent&McWilliams parametrization is worth studying.
- Isopycnal coordinates are interesting for theoretical analyses.
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Thank you for your attention