The hydrostatic approximation for stratified fluids

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Joint work with Roberta Bianchini (CNR, Rome)

Outline

The initial set of equations

Motivated by geophysical flows, we consider the heterogeneous incompressible Euler equations with gravity force

$$(E) \qquad \begin{cases} \partial_t \rho + \partial_x(\rho u) + \partial_z(\rho w) = 0, \\ \rho\big(\partial_t u + u\partial_x u + w\partial_z u\big) + \partial_x P = 0, \\ \rho\big(\partial_t w + u\partial_x w + w\partial_z w\big) + \partial_z P + g\,\rho = 0, \\ \partial_x u + \partial_z w = 0, \\ \text{boundary conditions.} \end{cases}$$

The velocity field is denoted $(u, w) : \Omega \to \mathbb{R} \times \mathbb{R}$, the density $\rho : \Omega \to \mathbb{R}_\star^+$.

Reconstructing the pressure $P : \Omega \to \mathbb{R}$ requires to solve an elliptic problem. Using instead the hydrostatic approximation, the system becomes ...

The hydrostatic approximation

The heterogeneous, incompressible Euler equations with gravity force

$$\text{(E)} \qquad \begin{cases} \partial_t \rho + \partial_x(\rho u) + \partial_z(\rho w) = 0, \\ \rho\big(\partial_t u + u\partial_x u + w\partial_z u\big) + \partial_x P = 0, \\ \rho\big(\partial_t w + u\partial_x w + w\partial_z w\big) + \partial_z P + g\,\rho = 0, \\ \partial_x u + \partial_z w = 0, \\ \text{boundary conditions.} \end{cases}$$

Using the hydrostatic approximation, the system becomes

$$\text{(H)} \qquad \begin{cases} \partial_t \rho + \partial_x(\rho u) + \partial_z(\rho w) = 0, \\ \rho\big(\partial_t u + u\partial_x u + w\partial_z u\big) + \partial_x P = 0, \\ \partial_z P = -g\rho, \qquad \Longleftrightarrow \qquad P = P|_{z=z_{\rm surf}} + g\int_{\cdot}^{z_{\rm surf}} \rho \, \mathrm{d}z \\ \partial_z w = -\partial_x u, \qquad \Longleftrightarrow \qquad w = w|_{z=z_{\rm bot}} - \int_{z_{\rm bot}}^{\cdot} \partial_x u \, \mathrm{d}z \\ \text{boundary conditions.} \end{cases}$$

Pb: stability of shear flows $(\rho, u)(t,x,z) = (\underline{\rho}(z), \underline{u}(z))$?

Some stability results

Homogeneous case: $\rho \equiv 1$.

- Spectral stability of the linearized system about shear flows, under the Rayleigh criterion $\underline{u}''(z) \neq 0$. [Rayleigh (1880)]
- Lyapunov stability under the Rayleigh criterion. [Arnold '65]
- Well-posedness of the (nonlinear) system in Sobolev spaces under the Rayleigh criterion. [Grenier '99], [Brenier '03], [Masmoudi&Wong '12].

Inhomogeneous case: $\partial_z \rho \neq 0$.

- Spectral stability of the linearized system about shear flows, under the Miles and Howard criterion $\frac{1}{4}|\underline{u}'(z)|^2 \leq \frac{-\underline{\rho}'(z)}{\underline{\rho}(z)}$. [Miles '61][Howard '61]
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Open problem

Well-posedness of the initial-value problem for the (linear or nonlinear) system?

Possible source of instabilities

The hydrostatic limit is singular:

$$(\mathrm{H}) \quad \begin{cases} \partial_t \rho + u\partial_x \rho + w\partial_z \rho = 0, \\ \rho\big(\partial_t u + u\partial_x u + w\partial_z u\big) + \partial_x P = 0, \\ \partial_z P = -g\rho, & \iff \quad P = P|_{z=z_{\rm surf}} + g\int_{\cdot}^{z_{\rm surf}} \rho \, \mathrm{d}z \\ \partial_z w = -\partial_x u, & \iff \quad w = w|_{z=z_{\rm bot}} - \int_{z_{\rm bot}}^{\cdot} \partial_x u \, \mathrm{d}z \\ \text{boundary conditions.} \end{cases}$$

- Contributions from $w\partial_z\rho$ and $\partial_x P$ compensate in energy estimates if $\rho > 0$, $\partial_z\rho < 0$. Stable stratification helps.
- There is no obvious way to deal with the contribution $w\partial_z u$. Shear velocity hurts.
- In the literature, theoretical results either
  1. Use the analytic framework. e.g. [Kukavica, Temam, Vicol & Ziane '11]
  2. Add regularization (viscosity, diffusivity). e.g. [Cao, Li & Titi '07+...]
  3. Relax the hydrostatic approximation [Desjardins, Lannes & Saut '21]

Eddy viscosity and diffusivity

We add to the system the contributions of non-resolved eddies suggested by [Gent & McWilliams '90]. They act as additional "bolus" velocities.

$$(\mathrm{H}_\kappa) \qquad \begin{cases} \partial_t \rho + (u + u_\star)\partial_x \rho + (w + w_\star)\partial_z \rho = 0, \\ \rho\big(\partial_t u + (u + u_\star)\partial_x u + (w + w_\star)\partial_z u\big) + \partial_x P = 0, \\ \partial_z P = -g\rho, \quad \partial_z w = -\partial_x u, \\ \text{boundary conditions (free surface).} \end{cases}$$

with

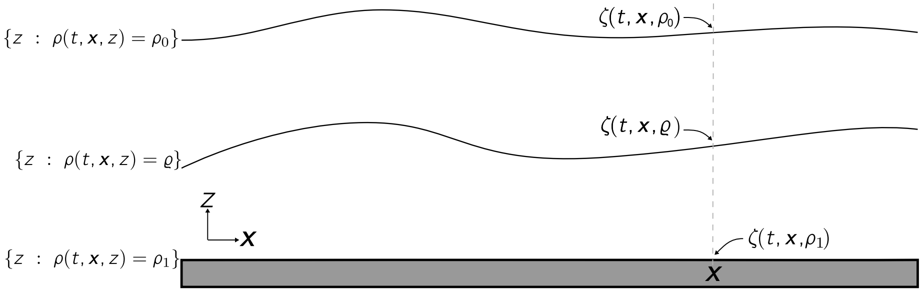
$$u_\star = \kappa \partial_z \left(\frac{\partial_x \rho}{\partial_z \rho} \right) , \quad w_\star = -\kappa \partial_x \left(\frac{\partial_x \rho}{\partial_z \rho} \right) ,$$

where $\kappa > 0$.

Isopycnal coordinates

We consider stratified flows and define the variable $h(t,x,\varrho) > 0$ through

$$h \overset{\text{def}}{=} -\partial_\varrho \zeta, \qquad \rho(t,x,\zeta(t,x,\varrho)) = \varrho, \quad \zeta(t,x,\rho(t,x,z)) = z.$$



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The system

$$(\mathrm{H}_\kappa) \qquad \begin{cases} \partial_t \rho + (u + u_\star)\partial_x \rho + (w + w_\star)\partial_z \rho = 0, \\ \rho\big(\partial_t u + (u + u_\star)\partial_x u + (w + w_\star)\partial_z u\big) + \partial_x P = 0, \\ \partial_z P = -g\rho, \quad \partial_z w = -\partial_x u, \\ \text{boundary conditions (free surface).} \end{cases}$$

reads in isopycnal coordinates

$$(\mathrm{H}_\kappa) \qquad \begin{cases} \partial_t h + \partial_x\big(h(u + u_\star)\big) = 0, \\ \partial_t u + (u + u_\star)\partial_x u + \partial_x\big(\mathcal{M} h\big) = 0, \end{cases}$$

where $u_\star \stackrel{\text{def}}{=} \kappa \frac{-\partial_x h}{h}$ and

$$\big(\mathcal{M} h\big)(t, x, \varrho) \stackrel{\text{def}}{=} g \int_{\rho_0}^{\rho_1} \frac{\min(\varrho, \varrho')}{\varrho} h(t, x, \varrho')\, \mathrm{d}\varrho'$$

The system in isopycnal coordinates

The isopycnal formulation of the hydrostatic system is

$$(\mathrm{H}_\kappa) \qquad \begin{cases} \partial_t h + u\partial_x h + h\partial_x u = \kappa\partial_x^2 h, \\ \partial_t u + \left(u + \kappa\frac{-\partial_x h}{h}\right)\partial_x u + \partial_x\left(\mathcal{M}h\right) = 0, \end{cases}$$

where $\mathcal{M}h(t,x,\varrho) \stackrel{\text{def}}{=} g\int_{\rho_0}^{\rho_1} \frac{\min(\varrho,\varrho')}{\varrho} h(t,x,\varrho')\,\mathrm{d}\varrho'$.

Remarks:

- The parameterization of [Gent & McWilliams '90] is nice and simple.
- The advection in the variable z (or ϱ) has disappeared.
- The domain is flattened.
- The system is easily discretized (multilayer framework) [Adim]
- The structure of the system is clarified:

$$\partial_t \begin{pmatrix} h \\ u \end{pmatrix} + \begin{pmatrix} 0 & \mathrm{Diag}(h) \\ \mathcal{M} & 0 \end{pmatrix} \partial_x \begin{pmatrix} h \\ u \end{pmatrix} + \begin{pmatrix} \mathrm{Diag}(u) & 0 \\ 0 & \mathrm{Diag}(u) \end{pmatrix} \partial_x \begin{pmatrix} h \\ u \end{pmatrix} = \kappa \begin{pmatrix} \partial_x^2 h \\ \frac{\partial_x h}{h} \end{pmatrix}$$

Structure of the system

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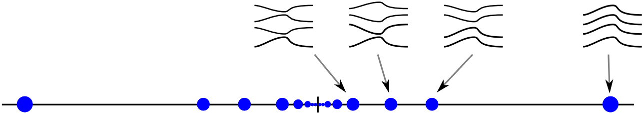
1. The first two terms generate wave propagation



2. Advection contribution mixes modes when $\partial_\varrho u \neq 0$.
3. The Gent&McWilliams contribution provides (partial) diffusion.[1]

[1]On a side note, if we define $v = u - \kappa \frac{\partial_x h}{h}$, then the system becomes

$$\partial_t \begin{pmatrix} h \\ v \end{pmatrix} + \begin{pmatrix} 0 & \mathrm{Diag}(h) \\ \mathcal{M} & 0 \end{pmatrix} \partial_x \begin{pmatrix} h \\ v \end{pmatrix} + \begin{pmatrix} \mathrm{Diag}(v) & 0 \\ 0 & \mathrm{Diag}(v - \kappa \frac{\partial_x h}{h}) \end{pmatrix} \partial_x \begin{pmatrix} h \\ v \end{pmatrix} = \kappa \begin{pmatrix} 0 \\ \partial_x^2 v \end{pmatrix},$$

and the Gent&McWilliams contribution acts as an effective viscosity on the variable v.

Some results

[R. Bianchini & VD]

For sufficiently regular data satisfying the stable stratification assumption

$$h\big|_{t=0} \geq h_0 > 0$$

and for any $\kappa > 0$, there exists a unique (classical) solution which we control on the time interval $[0, T]$ with

$$T^{-1} = C\left(1 + \kappa^{-1}\left(\left|\underline{u}'\right|_{L^2_\varrho}^2 + M_0^2\right)\right),$$

where M_0 is the size of the initial deviation from the shear flow equilibrium $(\underline{\rho}(\varrho), \underline{u}(\varrho))$, and C depends only on M_0, h_0 and the size of $(\underline{\rho}(\varrho), \underline{u}(\varrho))$.

[R. Bianchini & VD]

As long as this solution is controlled, we have strong convergence of the corresponding solutions to the incompressible Euler equations towards this solution in the limit of shallow water aspect ratio $L_z/L_x \ll 1$.

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Thoughts to go

- We do not understand well the stability of the hydrostatic equations. Stable stratification helps. Shear velocity hurts.
- The Gent&McWilliams parametrization is worth studying.
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Thank you for your attention