Perfectly Matched Layers for Mixed Hyperbolic-Dispersive equations

par

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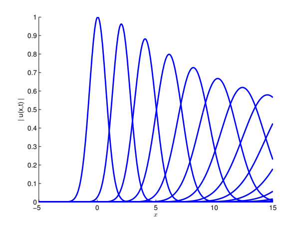
Journées Modélisation des Vagues à Phases Résolues

Linear Schrodinger equation

$$i\partial_t \psi + \Delta\psi + V(x,t,\psi)\,\psi = 0.$$



Time evolution of $|\psi|$

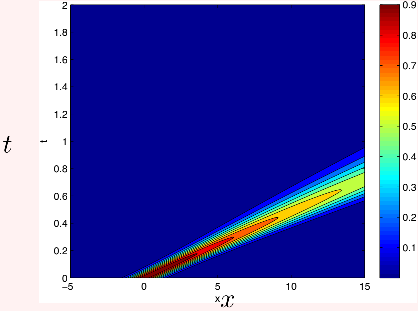
$$\Delta t = 0.2$$

Potential $V(x) = x$

Initial Data:

$\psi_0(x) = e^{-x^2+10ix}$

Exact solution



Linear Schrodinger equation

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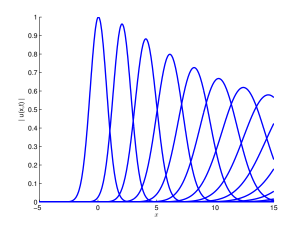

Time evolution of $|\psi|$

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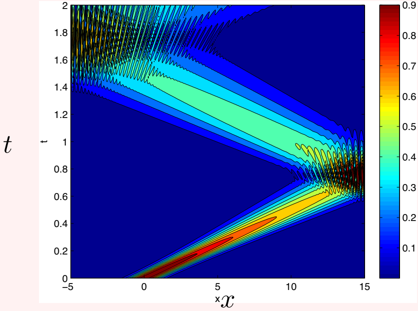
Numerical solution



Homogeneous Dirichlet conditions:

$$\psi_{|\Sigma} = 0$$

$\Rightarrow$ Unphysical reflections

Korteweg-de Vries (KdV) equation in $\mathbb{R}$

$$(KdV)\quad \begin{cases} \partial_t u + \partial_x u + \frac{3\varepsilon}{2} u\partial_x u + \frac{\mu}{6}\partial_x^3 u = 0, & (x,t) \in \mathbb{R} \times [0;T] \\ \lim_{|x|\to+\infty} u(x,t) = 0, & t \in [0;T] \\ u(x,0) = u_0(x), & x \in \mathbb{R} \end{cases}$$

Benjamin-Bona-Mahony (BBM) equation in $\mathbb{R}$

$$(BBM)\quad \begin{cases} (1 - \frac{\mu}{6}\partial_x^2)\partial_t u + \partial_x u + \frac{3\varepsilon}{2} u\partial_x u = 0, & (x,t) \in \mathbb{R} \times [0;T] \\ \lim_{|x|\to+\infty} u(x,t) = 0, & t \in [0;T] \\ u(x,0) = u_0(x), & x \in \mathbb{R} \end{cases}$$

- $u(x,t)$: real function
- u_0 has support compact in Ω
- unidirectional propagation of weakly nonlinear water-waves.
- μ : dispersion parameter, ε : amplitude parameter

- Phase velocity and group velocity of KdV equation are unbounded: not consistent with water wave propagation
- Hyperbolic version of water wave models (S Gavrilyuk and co-authors).

Example: Hyperbolic version of KdV (H-KdV) equation in $\mathbb{R}$

$$(HKdV) \quad \begin{cases} \partial_t u + u\partial_x u + \mu \partial_x \psi = 0, & \\ \partial_t \psi + \dfrac{\partial_x u - p}{\tau} = 0, \quad \partial_t p - \dfrac{\partial_x p - \psi}{\tau} = 0. & (x,t) \in \mathbb{R} \times [0;T] \\ \lim\limits_{|x| \to +\infty} |u(x,t)| + |\psi(x,t)| + |p(x,t)| = 0, & t \in [0;T] \\ u(x,0) = u_0(x), \psi(x,0) = \psi_0(x), p(x,0) = p_0(x) & x \in \mathbb{R} \end{cases}$$

Formally, as τ (relaxation parameter) goes to 0, one has:

$$p = \partial_x u + \tau \partial_{txx} u + O(\tau^2), \quad \psi = \partial_{xx} u + \tau \left(\partial_{txxx} u - \partial_{tx} u\right) + O(\tau^2),$$

$$\partial_t \left(u - \tau \partial_{xx} u + \tau \partial_{xxxx} u\right) + u\partial_x u + \mu \partial_{xxx} u = O(\tau^2).$$

- V. Duchene: *Rigorous justification of the Favrie-Gavrilyuk approximation to the Serre-Green-Naghdi model*, Nonlinearity (2019)

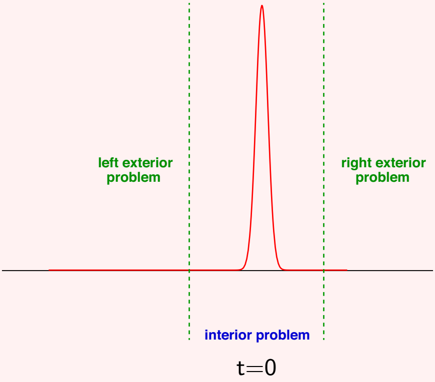
- KdV or BBM equations model "one way/right going"-water wave and discard the other "left-going" wave.
- Bona, Chen and Saut considered Boussinesq-BBM type equations:

Mixed Boussinesq-BMM equation in $\mathbb{R}$

$$(HKdV)\quad \begin{cases} (1-b\partial_{xx})\partial_t \eta + \partial_x u + a\partial_{xxx} u = 0, & \\ (1-d\partial_{xx})\partial_t u + \partial_x \eta + c\partial_{xxx}\eta = 0. & (x,t)\in\mathbb{R}\times[0;T] \\ \lim\limits_{|x|\to+\infty} |u(x,t)| + |\eta(x,t)| = 0, & t\in[0;T] \\ u(x,0) = u_0(x), \eta(x,0) = \eta_0(x), & x\in\mathbb{R} \end{cases}$$

- u_0, η_0 have compact support in an interval $[x_\ell, x_r]$.

$$\partial_t u + \partial_x^3 u = 0, \quad \text{sur } \mathbb{R} \times [0;T].$$



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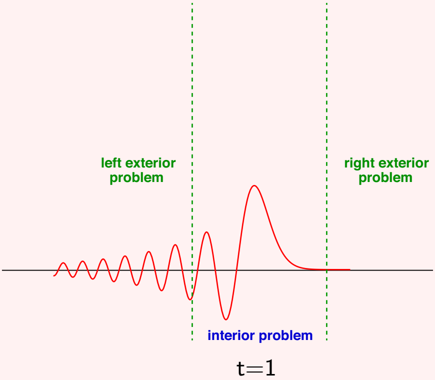

BC are different on left and right (odd number of BC)

1. Split the original eqs into a coupled eqs on interior and exterior domains
2. Apply the Laplace transform in time t
3. Solve the ODEs in x
4. Select finite energy solution (decreasing $|x| \to \pm\infty$)
5. Identify Dirichlet and Neumann data at $x_{l,r}$
6. Inverse Laplace transform

Airy equation set in $\Omega \in (x_\ell, x_r)$

$$\begin{cases} \partial_t u + \partial_x^3 u = 0, & (t,x) \in (0,T) \times (x_\ell, x_r), \\ u(0,x) = u_0(x),\ \mathrm{supp}(u_0) \in \Omega, & x \in (x_\ell, x_r), \\ \partial_t^{2/3} u - \partial_t^{1/3} \partial_x u + \partial_x^2 u = 0, & (t,x) \in (0,T) \times x_\ell, \\ \partial_t^{1/3} u + \partial_x u = 0,\ \partial_t^{2/3} u - \partial_x^2 u = 0, & (t,x) \in (0,T) \times \{x_r\}. \end{cases}$$

- Applications: linear equations (KdV, BBM , Schrodinger), nonlinear equations (mostly Schrodinger and nonlinear wave eqs) through paradifferential calculus
- Nonlocal BC (Memorize the solution): Padé approximant to get local BC
- $2D$-problems: issues in the corner (coupling two stable BC in two orthogonal half plane may be unstable)

Strategy II: Perfectly Matched Layers (after Berenger 1994) from P. Joly lecture notes

- Add an artificial absorbing layer around the computational domain
- "Perfectly Matched": waves do not reflect at the interface between PML and non-PML domains
- Principle (absorption in the x direction): transformation $\partial_x \to (1 + i\frac{\sigma(x)}{\omega})^{-1}\partial_x$ (in the frequency domain)

Example: 2D transport equation

$$\partial_t U + \mathbf{V_x}\partial_x U + \mathbf{V_y}\partial_y U = 0$$

PML counterpart in the x-direction: splitting $U = U^x + U^y$

$$\partial_t U^x + \sigma(x)U^x + \mathbf{V_x}\partial_x(U^x + U^y) = 0, \quad \partial_t U^y + \mathbf{V_y}\partial_y(U^x + U^y) = 0$$

- Systematic derivation and easy implementation
- Treatment of corners is simple
- Main drawbacks: mathematical not completely understood and examples of **instabilities**.

Outline

Linearized KdV equation in the frequency domain

$$-i\omega u + \mathbf{U}\partial_x u + \varepsilon \partial_{xxx} u = 0.$$

- Transformation $\partial_x \to (1 + i\frac{\sigma(x)}{\omega})^{-1}\partial_x$ and auxiliary functions

$$\partial_x u = (1 + \frac{i\sigma}{\omega})u_1, \quad \partial_x u_1 = (1 + \frac{i\sigma}{\omega})u_2.$$

PML equation for the KdV equation

$$\partial_t u + \sigma u + U\partial_x u + \varepsilon \partial_x u_2 = 0,$$
$$\partial_t(u_1 - \partial_x u) + \sigma u_1 = 0, \quad \partial_t(u_2 - \partial_x u_1) + \sigma u_2 = 0$$

Stability of PML equations:

1. If $U = 0$, the PML equations are always unstable.
2. If $\varepsilon U < 0$, the PML equations are stable if and only if $|\varepsilon|k^2 \geq 16|U|$.
3. If $\varepsilon U > 0$, the PML equations are stable if and only if $3|\varepsilon|k^2 \leq |U|$.

- **Remark:** a necessary stability condition (classical for PML):

$$v_g(k)v_\varphi(k) = (U - 3\varepsilon k^2)(U - \varepsilon k^2) > 0.$$

- $U = 0.4$, $I_{ref} = [-5, 5]$, $\sigma(x) = \sigma_0(\max(0, x - 5)^4 + \max(0, -x - 5)^4)$
- $u_0(x) = \exp(-40(x + 3)^2)$, $\delta x = 0.05$ $\delta t = \delta x$ $(CFL = 1)$
- Finite centered difference method and Crank-Nicolson scheme in time.
- Discretized equations stable if $\varepsilon < \varepsilon_c = \frac{U\delta x^2}{3}$ and unstable if $\varepsilon > \varepsilon_c$ or $\varepsilon U < 0$.

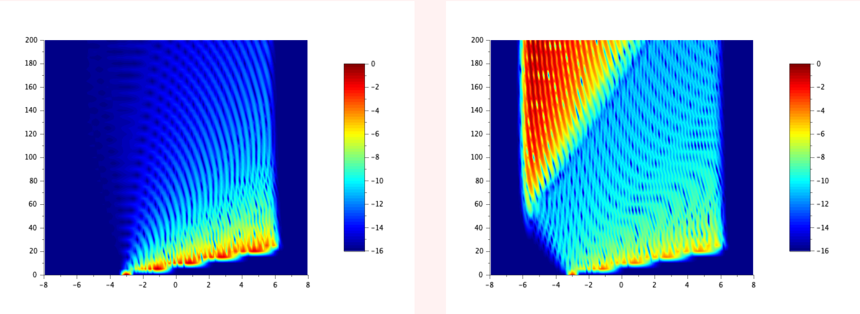

FIGURE: Representation of the function $v(t, x) = \log(1 + 1000|u(t, x)|)$ in the (x, t) plane $[-8, 8] \times [0, 200]$. On the left (stable case): $\varepsilon = U\delta x^2/4$. The solution is the Airy solution advected on the right with speed U. The outgoing waves are damped in the absorbing layers. On the right (unstable case): $\varepsilon = U\delta x^2/4$. The left going waves grow exponentially fast in the "absorbing" layer

Outline

Relaxation of KdV equation

$$\partial_t u + u\,\partial_x u + \mu\partial_x\psi = 0, \quad \partial_t p - \frac{\partial_x p - \psi}{\tau} = 0, \quad \partial_t\psi + \frac{\partial_x u - p}{\tau} = 0. \qquad (1)$$

- Dispersion relation: $k^2 = \dfrac{U - c}{(1 + \tau c)(\mu + \tau cU - \tau c^2)}$
- $c_{kdv}(k) \in]\frac{U}{2} - \sqrt{\frac{U^2}{4} + \frac{\mu}{\tau}}, U]$ and two velocities in $] - \infty, -\frac{1}{\tau}] \cup]\frac{U}{2} + \sqrt{\frac{U^2}{4} + \frac{\mu}{\tau}}, U]$

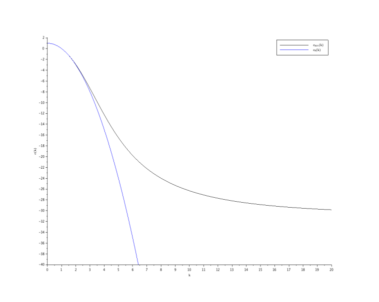

Figure: Dispersion relation for linearized KdV equation and system (1) (blue: KdV)

- Characteristic velocities:

$$\lambda_{\pm}(u) = \frac{u}{2} \pm \sqrt{\frac{u^2}{4} + \frac{\mu}{\tau}},\ \lambda_0 = -\frac{1}{\tau}.$$

- Riemann Invariants:

$$\psi + \int^u \lambda_{\pm}(s)ds = \psi + \frac{u}{2}\lambda_{\pm} \pm \frac{\ln(\lambda_+)}{\tau},\ p. \tag{2}$$

- An additional conservation law:

$$\left(\frac{u^2}{2\tau} + \mu\frac{p^2}{2} + \mu\frac{\psi^2}{2}\right)_t + \left(\frac{u^3}{3\tau} + \frac{\mu}{\tau}\psi u - \frac{\mu p^2}{2\tau}\right)_x = 0. \tag{3}$$

- even a Lagrangian formulation....

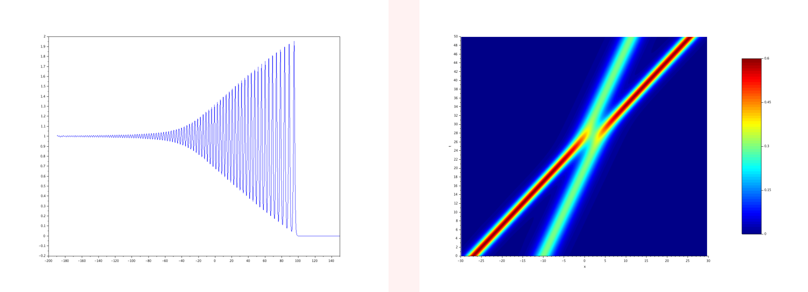

Figure: On the left: dispersive shock with step 1. On the right: A two soliton solution

Questions:

- Does hyperbolization-relaxation helps for PML?
- Nonlinear TBC through Riemann Invariants (work in progress with S. Gavrilyuk)?

Full PML equations

$$\partial_t u + \sigma u + U\partial_x u + \mu\partial_x\psi = 0, \quad \partial_t p + \sigma p - \frac{\partial_x p - \psi}{\tau} + \frac{\sigma}{\tau}\phi = 0,$$
$$\partial_t \psi + \sigma\psi + \frac{\partial_x u - p}{\tau} - \frac{\sigma}{\tau} q = 0, \quad \partial_t \phi = \psi, \quad \partial_t q = p.$$

PML for the first order system (not PM for the full system)

$$\partial_t u + \sigma u + U\partial_x u + \mu\partial_x\psi = 0, \quad \partial_t p + \sigma p - \frac{\partial_x p - \psi}{\tau} = 0,$$
$$\partial_t \psi + \sigma\psi + \frac{\partial_x u - p}{\tau} = 0.$$

- The later system admits a conservation law with a damping term:

$$\partial_t\left(\frac{u^2}{2\tau} + \mu\frac{p^2+\psi^2}{2}\right) + \sigma\left(\frac{u^2}{\tau} + \mu(p^2+\psi^2)\right) + \partial_x\left(U\frac{u^2}{2\tau} + \frac{\mu}{\tau}\psi u - \frac{\mu p^2}{2\tau}\right) = 0.$$

- The first system always generates instabilities, the second one is stable but not perfectly matched.

- $U = 1$, $\varepsilon = 5\delta x^2$ (unstable case for the full PML system)
- Relaxation parameter $\tau = 10^{-6}$, CFL=0.3, $\delta x = 0.02$
- $u_0(x) = \exp(-40(x+2)^2)$

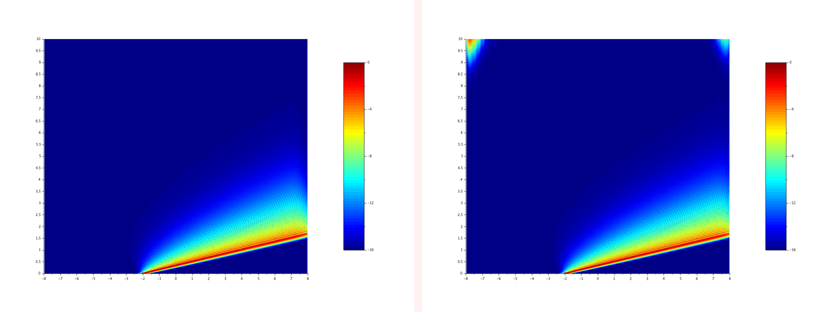

Figure: Representation of the function $v(t,x) = \log(1 + 1000|u(t,x)|)$ in the (x,t) plane $[-8,8] \times [0,10]$. On the left: partial "stable" PML conditions. On the right: complete "unstable" PML conditions. At time $t \approx 9$, a numerical instability occurs.

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Outline

We consider "bi-directional" models (both right and left going waves) introduced by Bona, Chen and Saut

BBM-Boussinesq eqs

$$
\begin{aligned}
(1 - b\partial_{xx})\partial_t \eta + \partial_x u + a\partial_{xxx} u = 0, \\
(1 - d\partial_{xx})\partial_t u + \partial_x u + c\partial_{xxx} \eta = 0
\end{aligned}
$$

Dispersion Relation

$$\omega_0^2(k) = k^2 \frac{(1 - ak^2)(1 - ck^2)}{(1 + bk^2)(1 + dk^2)}.$$

Some particular cases where the system is well-posed:

- Pure BBM-type system: $a = c = 0$, $b = d = 1/6$
- Pure KdV-type system: $b = d = 0$, $a = c = 1/6$
- Boussineq system (linearized Serre-Green-Naghdi eqs): $a = b = c = 0$ and $d = 1/3$.

Derivation of PML equations:

- $\partial_t \mapsto -i\omega$ and $\partial_x \mapsto (1 + \frac{i\sigma}{\omega})^{-1}\partial_x$
- Auxiliary functions $\eta_i = (1 + \frac{i\sigma}{\omega})^{-1}\partial_x \eta_{i-1}$, $u_i = (1 + \frac{i\sigma}{\omega})^{-1}\partial_x u_{i-1}$ for $i = 1, 2$.

PML EQS

$$\begin{aligned}
&\partial_t(\eta - b\eta_2) + \sigma(\eta - b\eta_2) + \partial_x(u + au_2) = 0,\\
&\partial_t(u - du_2) + \sigma(u - du_2) + \partial_x(\eta + c\eta_2) = 0,\\
&\partial_t(\eta_1 - \partial_x\eta) + \sigma\eta_1 = 0, \quad \partial_t(\eta_2 - \partial_x\eta_1) + \sigma\eta_2 = 0,\\
&\partial_t(u_1 - \partial_x u) + \sigma u_1 = 0, \quad \partial_t(u_2 - \partial_x u_1) + \sigma u_2 = 0.
\end{aligned} \tag{4}$$

- Dispersion relation: $k \mapsto (1 + \frac{i\sigma}{\omega})^{-1}k$ in the original dispersion
- **Necessary stability condition:** $\sigma \to 0$, roots bifurcating from $\pm w_0(k)$

$$v_\varphi(k)v_g(k) \geq 0, \quad v_\varphi(k) = \frac{\omega_0(k)}{k}, \quad v_g(k) = \frac{d\omega_0(k)}{dk}.$$

We can prove (linear) stability in the cases

1. Boussinesq equation: $a = b = c = 0$ and $d > 0$
2. Shallow water equations with surface tension: $a = b = d = 0$ and $c < 0$
3. BBM-KdV type: $a = d = 0$, $b > 0$, $c < 0$ or $b = c = 0$, $d > 0$, $a < 0$.

Arguments:

- Asymptotic expansion of solutions to the dispersion relation as $\sigma \to 0$
- No crossing argument to prove that the imaginary part $\mathrm{Im}(\omega) \leq 0$ for all $\sigma > 0$.

- Boussinesq equation: $a = b = c = 0$ and $d = 1/3$
- Domain of interest $[-6, 6]$. PML Domain $[-10, 10]$
- Absorption coefficient: $\sigma(x) = \max(x - 6, 0)^4 + \min(0, x + 6)^4$
- Hyperbolic right going wave: $\eta(0, x) = u(0, x) = \exp(-x^2)$. Dispersive right going wave: $u(0, x) = (1 - d\delta_{xx})^{-1/2}\eta(0, x)$

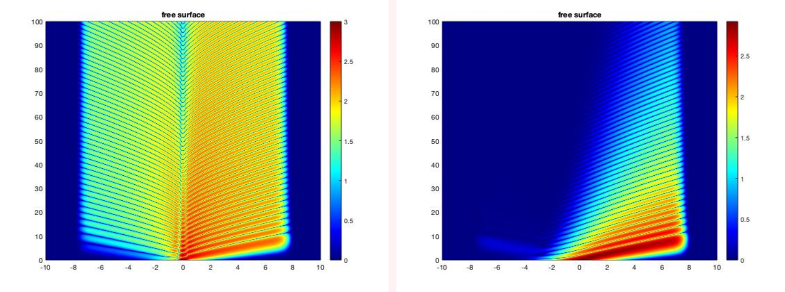

FIGURE: Unidirectional propagation: plots of $\log(1 + 1000|\eta(t, x)|)$ where $\eta(x, t)$ is the solution of Boussinesq eqs. On the left: "Hyperbolic" right going wave. There is a significant amount of the solution that propagates to the left. On the right: the initial condition is given by "dispersive" right going wave. The left-going part of the solution is negligible.

1. Full stability resultats for PML equations for KdV equation, hyperbolized version of KdV and Boussinesq eqs
2. PML is not suitable for KdV, partially for the hyperbolic version: hyperbolization does not help.
3. PML works for large class of BBM-Boussinesq equations
4. DTBC are better when $v_g(k)v_\varphi(k) < 0$ (which is a common situation in dispersive problems).

Future works:

1. Consider TBC for hyperbolic models with relaxation: either dissipative or dispersive like Favrie-Gavrilyuk model or LCT model (approximation of the Serre-Green-Naghdi equations)
2. Consider injection problems (in particular for hyperbolic equations with relaxation): impact of the order of the scheme (treatment of the ghost points)