Some advantages of considering dispersion as a constraint

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Journées de Modélisation
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Consider an **hyperbolic** model: $\partial_t \begin{pmatrix} H \\ U \end{pmatrix} + A \begin{pmatrix} H \\ U \end{pmatrix} \nabla \begin{pmatrix} H \\ U \end{pmatrix} = 0$ (hyperbolic)

with $H(t,x) \in \mathbb{R}^m$, $U(t,x) \in \mathbb{R}^n$ and $A \in M_{m+n}(\mathbb{R})$

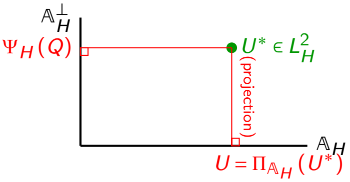
with an underlying **energy conservation** law: $\partial_t \left(\mathscr{E}(H) + \frac{1}{2} \langle U, U \rangle_H \right) = 0.$ (E)

The projected hyperbolic model (PH): reads [Kazolea, Parisot]

$$\partial_t \begin{pmatrix} H \\ U \end{pmatrix} + A \begin{pmatrix} H \\ U \end{pmatrix} \nabla \begin{pmatrix} H \\ U \end{pmatrix} = S \begin{pmatrix} H \\ U \end{pmatrix} - \begin{pmatrix} 0 \\ \Psi_H(Q) \end{pmatrix}, \quad \text{(S=Sources)} \quad \textit{bathymetry, friction, viscosity...}$$

$$L_H(U) = 0, \quad \text{(Contraint)} \quad U \in \mathbb{A}_H = \left\{ V \in L^2_H \mid L_H(V) = 0 \right\}$$

$$\langle V, \Psi_H(P) \rangle_H = 0, \quad \text{(Orthogonality)} \quad P \in \Psi_H^{-1}\left(\mathbb{A}_H^\perp \right)$$



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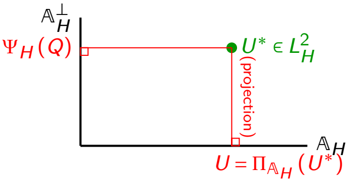

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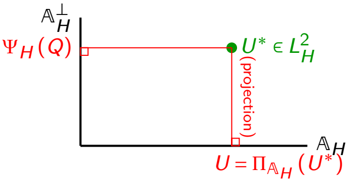

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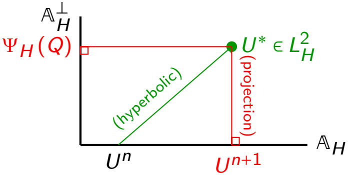


- Projection methods (hyperbolic)+(projection)
  [Chorin'68] [Parisot'19]
- Pseudo-compressiblility methods (hyperbolisation)
  [Chang, Kwak'84] [Favrie, Gavrilyuk'17]

1. Many **dispersive models of waves** (but not only!) satisfy this mathematical structure,
2. For any **linear** $L_H(U)$, the **energy conservation** law (E) still holds,
3. Similar to the **compressible structure**,

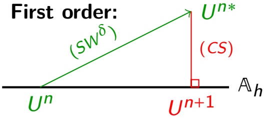
allows **reuse of tools** (analysis and numerical) from the literature.

Entropy-satisfying and high order schemes for (PH):

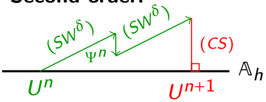
[Parisot'19]

Application of [Guermond, Minev, Shen'06]

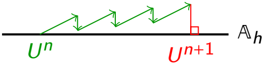
First order:

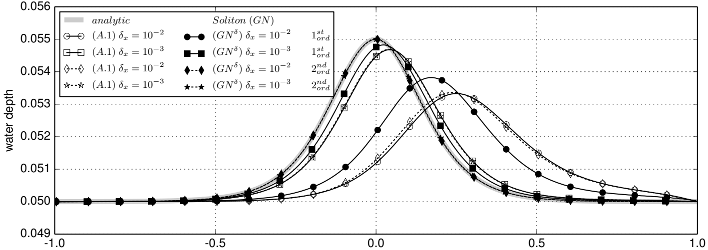


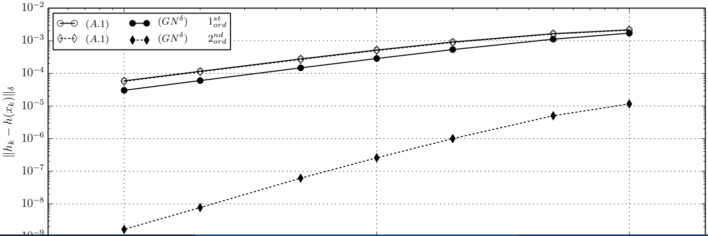
Second order:



Fourth order:







Consider the simple (PH) model: with $(c_1, c_2, \alpha) \in \mathbb{R}^3$

$$\begin{array}{rcl} H & = & \emptyset \\ U & = & (u, w) \end{array} , \; A = \begin{pmatrix} c_0 & 0 \\ 0 & c_1 \end{pmatrix} , \quad \begin{array}{rcl} L(v_0, v_1) & = & v_1 + \alpha \partial_x v_0 \\ \langle V, \widetilde{V} \rangle & = & \int_{\mathbb{R}} (v_0 \widetilde{v}_0 + v_1 \widetilde{v}_1) \, \mathrm{d}x \end{array} \quad \left| \quad \begin{array}{rcl} \partial_t u + c_0 \partial_x u & = & -\psi_0 \\ \partial_t w + c_1 \partial_x w & = & -\psi_1 \\ w & = & -\alpha \partial_x u \end{array} \right.$$

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- **Identification** of the dual space $\mathbb{A}^{\perp} = \{\Phi \mid \forall V \in \mathbb{A} \;, \langle V, \Phi \rangle = 0\}$:

$$0 = \langle V, \Phi \rangle = \int_{\mathbb{R}} (v_0 \phi_0 - \alpha \partial_x v_0 \phi_1) \,\mathrm{d}x = \int_{\mathbb{R}} v_0 (\phi_0 + \alpha \partial_x \phi_1) \,\mathrm{d}x \quad \Leftrightarrow \quad \boxed{\phi_0 = -\alpha \partial_x \phi_1}$$

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- **Identification** of the right-hand side $\Psi \in \mathbb{A}^\perp$:

$$\psi_0 = -\alpha\partial_x\psi_1 = \alpha\partial_x(\partial_t w + c_1\partial_x w) = -\alpha^2(\partial_{txx} u + \partial_x(c_1\partial_{xx} u))$$

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- **Replace** in the first equation :
$$\left(1 - \alpha^2 \partial_{xx}\right) \partial_t u + c_0 \partial_x u - \alpha^2 \partial_x (c_1 \partial_{xx} u) = 0$$

We recover the **KdV-BBM equation**.

The projected hyperbolic BBM model:

$$\begin{array}{rl} H &= \emptyset \\ U &= (u, w) \end{array}, \qquad A = \begin{pmatrix} \frac{u^{m-1}}{m} & 0 \\ 0 & c_1 \end{pmatrix}, \qquad \begin{array}{rl} L(v_0, v_1) &= v_1 + \partial_x v_0 \\ \langle V, \widetilde{V} \rangle &= \int_{\mathbb{R}} (v_0 \widetilde{v}_0 + v_1 \widetilde{v}_1)\, \mathrm{d}x \end{array}$$

is equivalent to the **mBBM** equation

$$\left(1 - \alpha^2 \partial_{xx}\right) \partial_t u + \partial_x u^m - \alpha^2 \partial_x (c_1 \partial_{xx} u) = 0.$$

The projected hyperbolic Camassa-Holm model:

$$\begin{array}{rcl} H &=& \emptyset \\ U &=& (u,w) \end{array}, \qquad A = \begin{pmatrix} 3u+2K & -w \\ 0 & u \end{pmatrix}, \qquad \begin{array}{rcl} L(v_0,v_1) &=& v_1 + \alpha\partial_x v_0 \\ \langle V, \widetilde{V}\rangle &=& \int_{\mathbb{R}} (v_0\widetilde{v}_0 + v_1\widetilde{v}_1)\,\mathrm{d}x \end{array}$$

is equivalent to the **Camassa-Holm** equation

$$\left(1-\alpha^2\partial_{xx}\right)\partial_t u + (3u+2K)\partial_x u - 2\alpha^2\partial_x u\partial_{xx} u - \alpha^2 u\partial_{xxx} u = 0.$$

The (*PH*) Serre/Green-Naghdi model:

[Fernández-Nieto, Parisot, Penel, Sainte-Marie'18]

$$\begin{array}{rcl} H &=& h \\ U &=& (u, w) \end{array}, \qquad A = \begin{pmatrix} u & h & 0 \\ g & u & 0 \\ 0 & 0 & u \end{pmatrix}, \qquad \begin{array}{rcl} L_H(v_0, v_1) &=& v_1 + \alpha h \partial_x v_0 \\ \langle V, \widetilde{V} \rangle_H &=& \int_{\mathbb{R}} (v_0 \widetilde{v}_0 + v_1 \widetilde{v}_1)\, h \,\mathrm{d}x \end{array}$$

is equivalent with $\alpha = \frac{1}{\sqrt{3}}$ to the **Serre/Green-Naghdi** equation [Lannes'13]

$$\begin{array}{rcl} \partial_t h + \partial_x (hu) &=& 0 \\ (1 + \mathscr{T}) \partial_t u + u \partial_x u &=& -g \partial_x h - \mathscr{Q}(v) \end{array}$$

with $\mathscr{T}(v) = -\frac{1}{3h} \partial_x \left(h^3 \partial_x v \right)$

and $\mathscr{Q}(v) = -\frac{1}{3h} \partial_x \left(h^3 v \partial_{xx} v - |\partial_x v|^2 \right)$

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The (almost)-(*PH*) Abbott model:

$$H = h, \quad U = (u,w), \qquad A = \begin{pmatrix} u & h & 0 \\ g & u & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \begin{aligned} L(v_0,v_1) &= v_1 + \alpha D \partial_x v_0 \\ \langle V, \widetilde{V} \rangle &= \int_{\mathbb{R}} (v_0 \widetilde{v}_0 + v_1 \widetilde{v}_1)\, D \,\mathrm{d}x \end{aligned}$$

is equivalent with $\alpha = \frac{1}{\sqrt{3}}$ to the **Abbott** equation.

⚠ **Not** the same scalar product $\Rightarrow$ **lost** of the energy conservation law

The (*PH*) Serre/Green-Naghdi model:

[Fernández-Nieto, Parisot, Penel, Sainte-Marie'18]

$$\begin{array}{rcl} H & = & h \\ U & = & (u, w, \sigma) \end{array}, \quad A = \begin{pmatrix} u & h & 0 & 0 \\ g & u & 0 & 0 \\ 0 & 0 & u & 0 \\ 0 & 0 & 0 & u \end{pmatrix}, \quad \begin{array}{rcl} L_H(v_0, v_1, v_2) & = & \begin{pmatrix} v_1 + \alpha h \partial_x v_0 - \beta v_0 \partial_x B \\ v_2 + \gamma h \partial_x v_0 \end{pmatrix} \\ \langle V, \widetilde{V} \rangle_H & = & \int_{\mathbb{R}} (v_0 \widetilde{v}_0 + v_1 \widetilde{v}_1 + v_2 \widetilde{v}_2) h \, \mathrm{d}x \end{array}$$

is equivalent with $\alpha = \frac{1}{2}$, $\beta = 1$ and $\gamma = \frac{1}{2\sqrt{3}}$ to the **Serre/Green-Naghdi** eq. [Lannes'13]

$$\begin{array}{rcl} \partial_t h + \partial_x (hu) & = & 0 \\ (1 + \mathcal{T}_b) \partial_t u + u \partial_x u & = & -g \partial_x h - \mathcal{Q}_b(v) \end{array}$$

with $\mathcal{T}_b(v) = -\frac{1}{3h} \partial_x \left(h^3 \partial_x v \right) + \frac{1}{2h} \left(\partial_x \left(h^2 v \partial_x B \right) - h^2 \partial_x B \partial_x v \right) + |\partial_x B|^2 v$

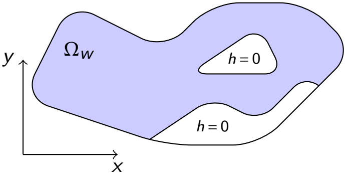
and $\mathcal{Q}_b(v) = -\frac{1}{3h} \partial_x \left(h^3 v \partial_{xx} v - |\partial_x v|^2 \right) + \frac{1}{2h} \left(\partial_x \left(h^2 v^2 \partial_{xx} B \right) - h^2 \left(v \partial_{xx} v - |\partial_x v|^2 \right) \partial_x B \right) + v^2 \partial_{xx} B \partial_x B$

[Noelle, Parisot, Tscherpel'22]

What about bounded domains?

- The L^2_h-scalar product can only be defined on the **wet domain**,

$$\Omega_w = \{x \in \Omega \mid h > 0\}.$$



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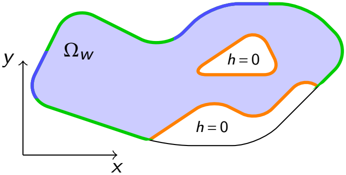

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- The L^2_h-scalar product can only be defined on the **wet domain**,
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- For any $V \in \mathbb{A}_h$ and $\Phi \in \mathbb{A}_h^\perp$, we have

$$0 = \langle V, \Phi \rangle_h = \int_{\partial\Omega_w} h^2\psi_1 \ u\cdot\mathfrak{n}\, d\chi = \int_{\Gamma_h} h^2\psi_1 \ u\cdot\mathfrak{n}\, d\chi + \int_{\Gamma_u} h^2\psi_1 \ u\cdot\mathfrak{n}\, d\chi + \int_{\Gamma_\psi} h^2\psi_1 \ u\cdot\mathfrak{n}\, d\chi$$

with $\Gamma_h = \{\chi \in \partial\Omega_w \mid h = 0\}$ (given).
We **must impose** $u\cdot\mathfrak{n}$ on $\Gamma_{\overline{u}}$ and $h^2\psi_1$ on Γ_ψ such that $\Gamma_u \cup \Gamma_\psi = \partial\Omega_w - \Gamma_h$.

Dry front + Wall	Fixed state	Outlet	Inlet
$\partial_x h = 0,\ u = 0$	$h = H,\ u = U$	$\partial_x h = 0,\ u = u^*$	$hu = M,\ h^2\phi_1 = HQ$

Coupling of (PH) model:

[Kazolea, Parisot]

	H, U	A	L_H
(GN)	$h, \begin{pmatrix} u \\ w \end{pmatrix}$	$\begin{pmatrix} u & h & 0 \\ g & u & 0 \\ 0 & 0 & u \end{pmatrix}$	$L_h^{\mathrm{GN}} = w + \frac{h}{\sqrt{3}} \nabla \cdot u$
(SW)	$h, \overline{u}$	$\begin{pmatrix} \overline{u} & h \\ g & \overline{u} \end{pmatrix}$	

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(SW)	$h, \begin{pmatrix} u \\ w \end{pmatrix}$	$\begin{pmatrix} u & h & 0 \\ g & u & 0 \\ 0 & 0 & u \end{pmatrix}$	$L_h^{\mathrm{SW}} = w$

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(SW)	$h, \begin{pmatrix} u \\ w \end{pmatrix}$	$\begin{pmatrix} u & h & 0 \\ g & u & 0 \\ 0 & 0 & u \end{pmatrix}$	$L_h^{\mathrm{SW}} = w$
Coupling (SW|GN) GN|SW SW|GN Adaptive	$h, \begin{pmatrix} u \\ w \end{pmatrix}$	$\begin{pmatrix} u & h & 0 \\ g & u & 0 \\ 0 & 0 & u \end{pmatrix}$	$L_h = w + \alpha(x) \frac{h}{\sqrt{3}} \nabla \cdot u$ $\alpha(x) = \begin{cases} 0 & \text{if } x \in \Omega^{\mathrm{SW}} \\ 1 & \text{if } x \in \Omega^{\mathrm{GN}} \end{cases}$

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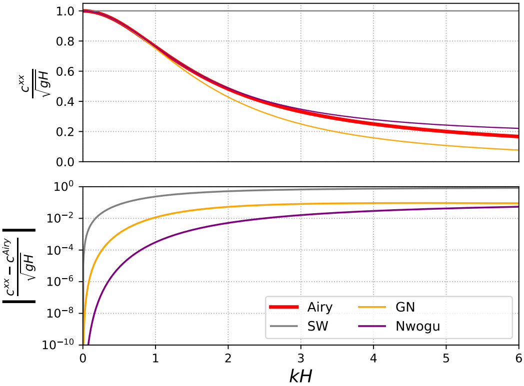
(GN)	$h, \begin{pmatrix} u \\ w \end{pmatrix}$	$\begin{pmatrix} u & h & 0 \\ g & u & 0 \\ 0 & 0 & u \end{pmatrix}$	$L_h^{\mathrm{GN}} = w + \frac{h}{\sqrt{3}} \nabla \cdot u$
(SW)	$h, \begin{pmatrix} u \\ w \end{pmatrix}$	$\begin{pmatrix} u & h & 0 \\ g & u & 0 \\ 0 & 0 & u \end{pmatrix}$	$L_h^{\mathrm{SW}} = w$
Coupling (SW|GN) GN|SW SW|GN Adaptive	$h, \begin{pmatrix} u \\ w \end{pmatrix}$	$\begin{pmatrix} u & h & 0 \\ g & u & 0 \\ 0 & 0 & u \end{pmatrix}$	$L_h = w + \alpha(x) \frac{h}{\sqrt{3}} \nabla \cdot u$ $\alpha(x) = \begin{cases} 0 & \text{if } x \in \Omega^{\mathrm{SW}} \\ 1 & \text{if } x \in \Omega^{\mathrm{GN}} \end{cases}$
Coupling (B|GN)	$h, \begin{pmatrix} u \\ w \end{pmatrix}$	$\begin{pmatrix} u & h & 0 \\ g & u & 0 \\ 0 & 0 & u \end{pmatrix}$	$L_h = w + (\alpha h + (1-\alpha) D) \nabla \cdot u$ $\alpha(x) = \begin{cases} 0 & \text{if } x \in \Omega^{\mathrm{B}} \\ 1 & \text{if } x \in \Omega^{\mathrm{GN}} \end{cases}$

Improved dispersion with (PH) models: [Kazolea, Parisot]

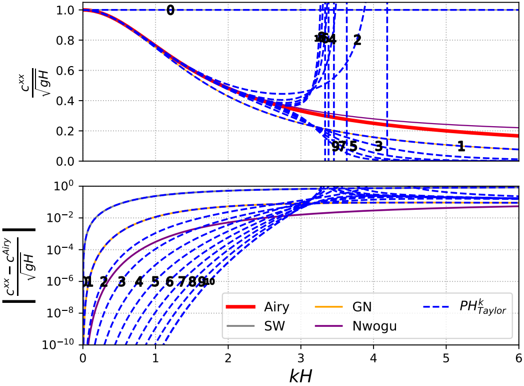
	H, U	A	L_H	$\frac{c^2}{gH}$
(SW)	$h, (u)$	$\begin{pmatrix} u & h \\ g & u \end{pmatrix}$		1
(GN)	$h, \begin{pmatrix} u \\ w \end{pmatrix}$	$\begin{pmatrix} u & h & 0 \\ g & u & 0 \\ 0 & 0 & u \end{pmatrix}$	$w + \frac{h}{\sqrt{3}} \partial_x u$	$\frac{1}{1+\frac{1}{3}\|kH\|^2}$



Improved dispersion with (PH) models:

[Kazolea, Parisot]

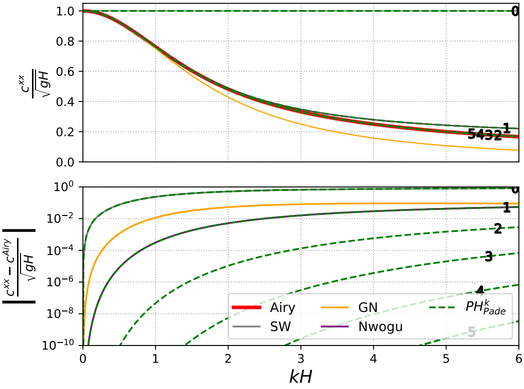
	H, U	A	L_H	$\frac{c^2}{gH}$
Taylor expansion [Madsen, Sørensen'92]	$h, \begin{pmatrix} u \\ w_{1\ldots N} \end{pmatrix}$	$\begin{pmatrix} u & h & 0 \\ g & u & \\ 0 & & u\ \mathrm{I}_N \end{pmatrix}$	$w_i - \alpha_i (-h)^i \partial_x^i u$	$\frac{1}{1+\sum_i \left\lvert \alpha_i \lvert kH \rvert^i \right\rvert^2}$



Improved dispersion with (PH) models:

[Kazolea, Parisot]

	H, U	A	L_H	$\frac{c^2}{gH}$
Taylor expansion [Madsen, Sørensen'92]	$h, \begin{pmatrix} u \\ w_{1\dots N} \end{pmatrix}$	$\left(\begin{array}{cc\|c} u & h & 0 \\ g & u & \\ \hline & 0 & u\,\mathrm{I}_N \end{array}\right)$	$w_i - \alpha_i (-h)^i \partial_x^i u$	$\frac{1}{1+\sum_i \left\|\alpha_i \|kH\|^i\right\|^2}$
Padé approximant [Nwogu'93] [Choi'22]	$h, \begin{pmatrix} u \\ w_{1\dots N} \\ \widetilde{w}_{1\dots M} \end{pmatrix}$	$\left(\begin{array}{cc\|c\|c} u & h & 0 & 0 \\ g & u & & \\ \hline & 0 & u & 0 \\ \hline & 0 & 0 & u \end{array}\right)$	$\overline{w}_i - \alpha_i (-h)^i \partial_x^i \overline{u}$ $\partial_t \widetilde{w}_i - \widetilde{\alpha}_i (-h)^i \partial_x^i \overline{u}$	$\frac{1-\sum_i \left\|\frac{\widetilde{\alpha}_i \|kH\|^i}{\sqrt{gH}}\right\|^2}{1+\sum_i \left\|\alpha_i \|kH\|^i\right\|^2}$



Improved dispersion with (PH) models:

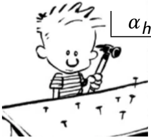
[Kazolea, Parisot]

	H, U	A	L_H	$\frac{c^2}{gH}$
Taylor expansion [Madsen, Sørensen'92]	$h, \begin{pmatrix} u \\ w_{1\dots N} \end{pmatrix}$	$\left(\begin{array}{cc\|c} u & h & 0 \\ g & u & \\ \hline \multicolumn{2}{c\|}{0} & u\,\mathrm{I}_N \end{array}\right)$	$w_i - \alpha_i (-h)^i \partial_x^i u$	$\frac{1}{1+\sum_i \left\| \alpha_i \|kH\|^i \right\|^2}$
Padé approximant [Nwogu'93] [Choi'22]	$h, \begin{pmatrix} u \\ w_{1\dots N} \\ \widetilde{w}_{1\dots M} \end{pmatrix}$	$\left(\begin{array}{cc\|c\|c} u & h & 0 & 0 \\ g & u & & \\ \hline \multicolumn{2}{c\|}{0} & u & 0 \\ \hline \multicolumn{2}{c\|}{0} & 0 & u \end{array}\right)$	$\overline{w}_i - \alpha_i (-h)^i \partial_x^i \overline{u}$ $\partial_t \widetilde{w}_i - \widetilde{\alpha}_i (-h)^i \partial_x^i \overline{u}$	$\frac{1-\sum_i \left\| \frac{\widetilde{\alpha}_i \|kH\|^i}{\sqrt{gH}} \right\|^2}{1+\sum_i \left\| \alpha_i \|kH\|^i \right\|^2}$
fully dispersive [Duchêne, Israwi, Talhouk'16]	$h, \begin{pmatrix} u \\ w \end{pmatrix}$	$\begin{pmatrix} u & h & 0 \\ g & u & 0 \\ 0 & 0 & u \end{pmatrix}$	$w + \alpha_h h \partial_x u$ $\alpha_h = \sqrt{\frac{\widetilde{k_h} - \tanh(\widetilde{k_h})}{\widetilde{k_h}^2 \tanh(\widetilde{k_h})}}$ $\widetilde{k_h}$: "the wave number"	$\frac{\tanh(kH)}{kH}$

α_h is a function of space, **not** of wave number.

Improved dispersion with (PH) models: [Kazolea, Parisot]

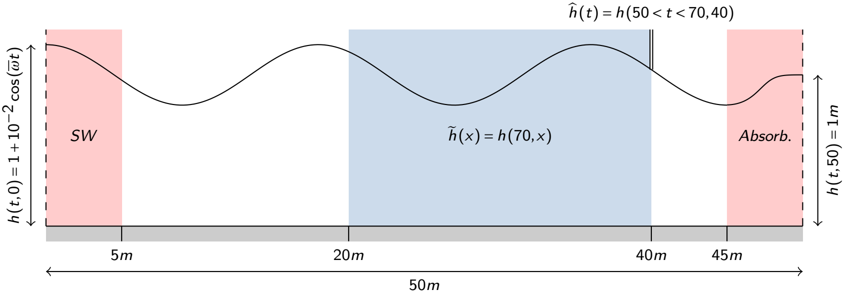
	H,U	A	L_H	$\frac{c^2}{gH}$
Taylor expansion [Madsen, Sørensen'92]	$h, \begin{pmatrix} u \\ w_{1\dots N} \end{pmatrix}$	$\left(\begin{array}{cc\|c} u & h & \\ g & u & 0 \\ \hline & 0 & u\,\mathrm{I}_N \end{array}\right)$	$w_i - \alpha_i (-h)^i \partial_x^i u$	$\frac{1}{1+\sum_i \left\|\alpha_i \|kH\|^i\right\|^2}$
Padé approximant [Nwogu'93] [Choi'22]	$h, \begin{pmatrix} u \\ w_{1\dots N} \\ \widetilde{w}_{1\dots M} \end{pmatrix}$	$\left(\begin{array}{cc\|c\|c} u & h & 0 & 0 \\ g & u & & \\ \hline & 0 & u & 0 \\ \hline & 0 & 0 & u \end{array}\right)$	$\overline{w}_i - \alpha_i (-h)^i \partial_x^i \overline{u}$ $\partial_t \widetilde{w}_i - \widetilde{\alpha}_i (-h)^i \partial_x^i \overline{u}$	$\frac{1-\sum_i \left\|\frac{\widetilde{\alpha}_i \|kH\|^i}{\sqrt{gH}}\right\|^2}{1+\sum_i \left\|\alpha_i \|kH\|^i\right\|^2}$
fully dispersive [Duchêne, Israwi, Talhouk'16]	$h, \begin{pmatrix} u \\ w \end{pmatrix}$	$\begin{pmatrix} u & h & 0 \\ g & u & 0 \\ 0 & 0 & u \end{pmatrix}$	$w + \alpha_h h \partial_x u$ $\alpha_h = \sqrt{\frac{\widetilde{k_h} - \tanh(\widetilde{k_h})}{\widetilde{k_h}^2 \tanh(\widetilde{k_h})}}$ $\widetilde{k_h}$: "the wave number"	$\frac{\tanh(kH)}{kH}$



α_h is a function of space, **not** of wave number.

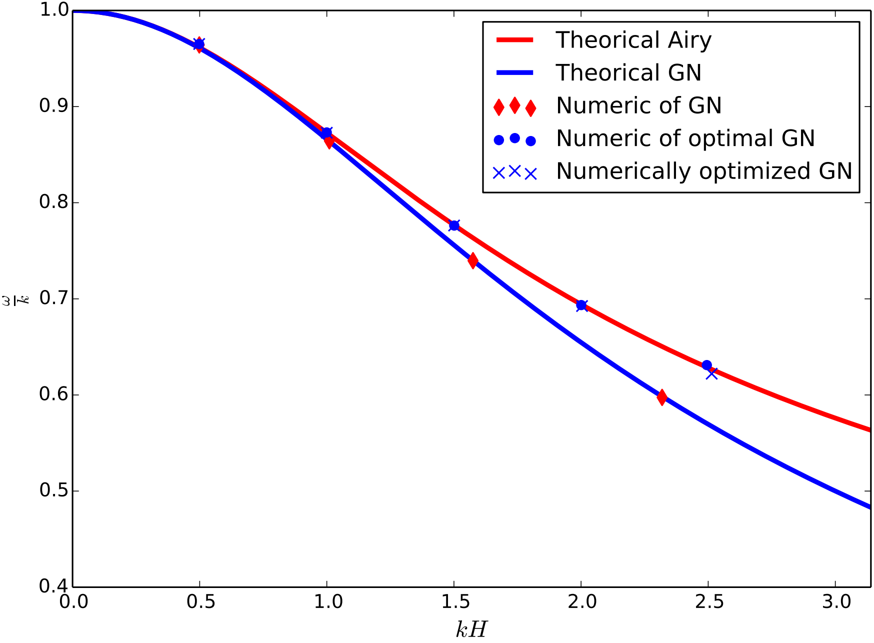
- **Pic wave number (WN)** and local FFT
- **Closer monochromatic WN**
- **Zero crossing WN**
- **Derivative approximation**

Dispersion test case: For a given $\overline{\omega}$



Using mean-square approximation, we fit

- $\widehat{h}(t) \approx 1 + \widehat{\eta} cos\left(\widehat{\phi} - \omega t\right)$
- $\widetilde{h}(x) \approx 1 + \widetilde{\eta} cos\left(kx - \widetilde{\phi}\right)$



Theorical Airy
Theorical GN
Numeric of GN
Numeric of optimal GN
Numerically optimized GN
1.0
0.9
0.8
0.7
0.6
0.5
0.4
$\frac{\omega}{k}$
0.0
0.5
1.0
1.5
2.0
2.5
3.0
kH

Advantages of (PH) for modeling:

- Models always satisfy an **energy conservation law**
- Give some information on the **boundary conditions**
- Allow easy **coupling** of (PH) models
- Allow **improvement** of properties
  dispersion relation, shoaling coefficient, ...
- Allow to include **more physics**
  H: salinity, temperature, ...
  U: shear/vorticity, ...

Advantages of (PH) for numerics:

- **Structure preserving** schemes
  entropy-satisfying, "curl"-free, ...
- **Well-Balanced** schemes
  preserve the moving steady state
- **Low cost high order** schemes
  only one implicite step
- Dispersion **adaptive** schemes
  solve dispersion only where and when it is useful
- **Parareal** schemes
  time parallelization (ask Lucas Perrin)

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Thank You