Boundary conditions for Boussinesq-type models in elevation-flux form

David Lannes$^1$, Mathieu Rigal$^1$

$^1$Institut de Mathématiques de Bordeaux

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Objectives

Postdoc supported by Institut des Mathématiques de la Planète Terre

Supervision: David Lannes and Philippe Bonneton

Long term goal: study extreme waves in littoral area

- Need accurate dispersive model: Boussinesq-type systems
- Boundary conditions are difficult to deal with

Recently: Perfectly Matched Layer, source function method → costly
Postdoc supported by **Institut des Mathématiques de la Planète Terre**

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Long term goal: study extreme waves in littoral area

- Need accurate dispersive model: **Boussinesq-type systems**
- Boundary conditions are difficult to deal with

*Recently*: Perfectly Matched Layer, source function method → costly

→ We propose a new and efficient method for boundary conditions.
Consider the Boussinesq-Abbott system

\[
\begin{aligned}
\partial_t \zeta + \partial_x q &= 0 \\
(1 - \kappa^2 \partial_{xx}^2) \partial_t q + \partial_x f_{\text{NSW}}(\zeta, q) &= 0
\end{aligned}
\]  

in \((0, \ell)\) (BA)

with generating boundary conditions

\[
\zeta(t, 0) = g_0(t), \quad \zeta(t, \ell) = g_\ell(t),
\]

where \(g_0, g_\ell \in C(0, T)\) and

\[
\kappa^2 = H_0^2 / 3, \quad f_{\text{NSW}}(\zeta, q) = hu^2 + gh^2 / 2
\]
Reformulation of the model

How to account for boundary conditions? How to recover $q_{|x=0,\ell}$?

- Hyperbolic case ($\kappa = 0$): Riemann invariants
- Dispersive case ($\kappa > 0$): need to invert $(1 - \kappa^2 \partial_{xx}) \rightarrow$ requires knowledge on $\partial_t q_{|x=0,\ell}$
  
  Lannes and Weynans 2020 “Generating boundary conditions for a Boussinesq system”
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Lannes and Weynans 2020 “Generating boundary conditions for a Boussinesq system”

Fix \( 0 \leq t \leq T \), then \( y(x) = \partial_t q(t, x) \) satisfies an ODE of the form

\[
\begin{cases}
    y - \kappa^2 y'' = \phi(x) \\
    y(0) = \dot{q}_0, \quad y(\ell) = \dot{q}_\ell
\end{cases}
\]

Equivalently: \( y = y_h + y_b \) with \( \begin{cases}
    y_h - \kappa^2 y_h'' = 0 \\
    y_h(0) = \dot{q}_0, \quad y_h(\ell) = \dot{q}_\ell
\end{cases} \) and \( \begin{cases}
    y_b - \kappa^2 y_b'' = \phi(x) \\
    y_b(0) = y_b(\ell) = 0
\end{cases} \)
Reformulation of the model

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- Hyperbolic case ($\kappa = 0$): Riemann invariants
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    y_h - \kappa^2 y_h''' = 0 \\
    y_h(0) = \dot{q}_0, \quad y_h(\ell) = \dot{q}_\ell
\end{cases}
\quad \text{and} \quad
\begin{cases}
    y_b - \kappa^2 y_b''' = \phi(x) \\
    y_b(0) = y_b(\ell) = 0
\end{cases}
$$

Define $R^0$ as the inverse of $(1 - \kappa^2 \partial_{xx}^2)$ with **homogeneous** Dirichlet conditions. Then

$$
\partial_t q = -R^0 \partial_x f_{\text{NSW}} + s_{(0)} \dot{q}_0 + s_{(\ell)} \dot{q}_\ell
$$

where

$$
\begin{cases}
    (1 - \kappa^2 \partial_x^2) s_{(0)} = 0 \\
    s_{(0)}(0) = 1, \quad s_{(0)}(\ell) = 0
\end{cases}
\quad \text{and} \quad
\begin{cases}
    (1 - \kappa^2 \partial_x^2) s_{(\ell)} = 0 \\
    s_{(\ell)}(0) = 0, \quad s_{(\ell)}(\ell) = 1
\end{cases}
$$

(2)
Reformulation of the model

Note $R^1$ the inverse of $(1 - \kappa^2 \partial^2_{xx})$ with **homogeneous Neumann** conditions.
Reformulation of the model

Note $R^1$ the inverse of $(1 - \kappa^2 \partial^2_{xx})$ with **homogeneous Neumann** conditions

### Proposition 1 (Equivalent formulation with nonlocal flux)

Let $(\zeta, q)$ initially equal to $(\zeta^{\text{in}}, q^{\text{in}})$. The two assertions are equivalent:

1. The pair $(\zeta, q)$ satisfies (BA) with generating conditions $\zeta(\cdot, 0) = g_0$ and $\zeta(\cdot, \ell) = g_\ell$
2. The pair $(\zeta, q)$ satisfies

\[
\begin{align*}
\frac{\partial}{\partial t} \zeta + \frac{\partial}{\partial x} q &= 0 \\
\frac{\partial}{\partial t} q + \frac{\partial}{\partial x} (R^1 f_{\text{NSW}}) &= \tilde{s}(0) \dot{q}_0 + \tilde{s}(\ell) \dot{q}_\ell
\end{align*}
\]

in $(0, \ell)$, \hspace{1cm} (3)

with

\[
\begin{pmatrix}
\tilde{s}'(0)(0) & \tilde{s}'(\ell)(0) \\
\tilde{s}'(0)(\ell) & \tilde{s}'(\ell)(\ell)
\end{pmatrix}
\begin{pmatrix}
\dot{q}_0 \\
\dot{q}_\ell
\end{pmatrix}
= \frac{1}{\kappa^2} \begin{pmatrix}
(R^1 - \text{id})_{00} f_{\text{NSW}} \\
(R^1 - \text{id})_{\ell\ell} f_{\text{NSW}}
\end{pmatrix} - \begin{pmatrix}
\ddot{g}_0 \\
\ddot{g}_\ell
\end{pmatrix}
\]

\hspace{1cm} (4)
Reformulation of the model

Note $R^1$ the inverse of $(1 - \kappa^2 \partial_{xx}^2)$ with **homogeneous Neumann** conditions

**Proposition 1 (Equivalent formulation with nonlocal flux)**

Let $(\zeta, q)$ initially equal to $(\zeta^{\text{in}}, q^{\text{in}})$. The two assertions are equivalent:

1. **The pair** $(\zeta, q)$ **satisfies (BA) with generating conditions** $\zeta(\cdot, 0) = g_0$ and $\zeta(\cdot, \ell) = g_\ell$

2. **The pair** $(\zeta, q)$ **satisfies**

\[
\begin{cases}
\partial_t \zeta + \partial_x q = 0 \\
\partial_t q + \partial_x (R^1 f_{\text{NSW}}) = \dot{s}_0 \dot{q}_0 + \dot{s}_\ell \dot{q}_\ell
\end{cases}
\]  
  \text{in } (0, \ell),

with

\[
\begin{pmatrix}
\dot{s}_0' (0) & \dot{s}_\ell' (0) \\
\dot{s}_0' (\ell) & \dot{s}_\ell' (\ell)
\end{pmatrix}
\begin{pmatrix}
\dot{q}_0 \\
\dot{q}_\ell
\end{pmatrix}
= \frac{1}{\kappa^2} \left( (R^1 - \text{id})_{|0} f_{\text{NSW}} \right) - \left( \ddot{g}_0 \right)
\]  

\text{in } (0, \ell),

**Sketch of the proof:**

- To get (3), check that $R^0 \partial_x = \partial_x R^1$.
- Apply $\partial_x$ to the discharge eq. from (3); take the trace at $x = 0, \ell$ to get (4).

\[
\begin{align*}
\partial_{xt}^2 q + (\partial_{xx} R^1 f_{\text{NSW}}) &= \dot{s}_0' \dot{q}_0 + \dot{s}_\ell' \dot{q}_\ell \\
-\partial_{tt}^2 \zeta &= \frac{1}{\kappa^2} (\text{id} - R^1) f_{\text{NSW}}
\end{align*}
\]
Other boundary conditions

Possibility to enforce general boundary conditions

\[ \xi^+[\zeta, q](t, 0) = g_0(t), \quad \xi^-[\zeta, q](t, \ell) = g_\ell(t). \]  

(5)

For instance, \( \xi^\pm \) given by \( q \) or Riemann invariants

\[ \mathcal{R}_\pm(U) = u \pm 2 \sqrt{gh}. \]
Other boundary conditions

Possibility to enforce general boundary conditions

$$\xi^+[\zeta, q](t, 0) = g_0(t), \quad \xi^-[\zeta, q](t, \ell) = g_\ell(t). \quad (5)$$

For instance, $\xi^\pm$ given by $q$ or Riemann invariants

$$R_\pm(U) = u \pm 2 \sqrt{gh}.$$

Adapt trace ODE in terms of missing data (outgoing information $\xi^-_0$ and $\xi^+_{\ell}$)

\[
\begin{pmatrix}
\xi^+_0 & \xi^-_0 \\
\xi^+_\ell & \xi^-_\ell
\end{pmatrix}
\begin{pmatrix}
\xi^+_0'(0) & \xi^-_0'(0) \\
\xi^+_\ell' & \xi^-_\ell'
\end{pmatrix}
\frac{d}{dt} q(\xi^\pm_0) = \frac{1}{\kappa^2} \left( (R^1 - \text{id})_{|0} f_{\text{NSW}} \right) - \frac{d^2}{dt^2} \left( \zeta(\xi^\pm_0) \right)
\]
Numerical scheme for the reformulated system

Discretize \((0, \ell)\) as follows:

\[
\begin{align*}
0 & \quad \Delta x & \quad \ell - \Delta x & \quad \ell \\
 x_1 & & x_2 & \cdots & x_{N-1} & x_N \\
\end{align*}
\]

\text{Time stepping procedure}

Step 1: Define \(\mathbf{R}_1 f_n\) \(\text{NSW}\) as the vector \(\mathbf{v} \in \mathbb{R}^N\) satisfying

\[
\begin{align*}
\mathbf{v}_i - \kappa_2 \mathbf{v}_{i+1} - 2 \mathbf{v}_i + \mathbf{v}_{i-1} & = f_n(U_{in}) & 2 \leq i \leq N - 1 \\
\mathbf{v}_2 - \mathbf{v}_1 \Delta x & = \mathbf{v}_N - \mathbf{v}_{N-1} \Delta x & = 0
\end{align*}
\]

Similar definition for \(s(0)\) and \(s(\ell)\).
Numerical scheme for the reformulated system

Discretize \((0, \ell)\) as follows:

\[
\begin{align*}
    &x_1 & &x_2 & & & &x_{N-1} & &x_N \\
    &0 & &\Delta x & & & &\ell - \Delta x & &\ell
\end{align*}
\]

Note \(U^n_i = (\zeta^n_i, q^n_i)^T\) the approximation of \(\frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} \begin{pmatrix} \zeta \\ q \end{pmatrix} (t^n, s) \, ds\).
Numerical scheme for the reformulated system

Discretize \((0, \ell)\) as follows:

\[
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&x_1 \quad x_2 \quad \ldots \quad x_{N-1} \quad x_N \\
&0 \quad \Delta x \quad \ell - \Delta x \quad \ell
\end{align*}
\]

Note \(U^n_i = (\zeta^n_i, q^n_i)^T\) the approximation of \(\frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} \begin{pmatrix} \zeta \\ q \end{pmatrix} (t^n, s) \, ds\).

Time stepping procedure

**Step 1:** Define \(R^1 f_{NSW}^n\) as the vector \(v \in \mathbb{R}^N\) satisfying

\[
\begin{cases}
    v_i - \kappa^2 \frac{v_{i+1} - 2v_i + v_{i-1}}{\Delta x^2} = f_{NSW}(U^n_i) \quad \text{for } 2 \leq i \leq N - 1 \\
    \frac{v_2 - v_1}{\Delta x} = \frac{v_N - v_{N-1}}{\Delta x} = 0
\end{cases}
\]

Similar definition for \(\xi(0)\) and \(\xi(\ell)\).
Time stepping procedure

Step 2: Approx. trace ODEs using FD scheme to get $\delta_t q_1^n, \delta_t q_N^n$; Update border values

\[
\begin{align*}
q_1^{n+1} &= q_1^n + \Delta t \delta_t q_1^n \\
q_N^{n+1} &= q_N^n + \Delta t \delta_t q_N^n \\
\end{align*}
\]

and

\[
\begin{align*}
\zeta_1^{n+1} &= g_0(t^{n+1}) \\
\zeta_N^{n+1} &= g_\ell(t^{n+1}) \\
\end{align*}
\]
Numerical scheme for the reformulated system

Time stepping procedure

Step 2: Approx. trace ODEs using FD scheme to get $\delta_t q^n_1, \delta_t q^n_N$; Update border values

\[
\begin{align*}
q^{n+1}_1 &= q^n_1 + \Delta t \delta_t q^n_1 \\
q^{n+1}_N &= q^n_N + \Delta t \delta_t q^n_N
\end{align*}
\]

and

\[
\begin{align*}
\zeta^{n+1}_1 &= g_0(t^{n+1}) \\
\zeta^{n+1}_N &= g_\ell(t^{n+1})
\end{align*}
\]

Step 3: For $2 \leq i \leq N$, finite volume update with Lax-Friedrichs numerical flux

\[
\begin{align*}
\frac{\zeta^{n+1}_i - \zeta^n_i}{\Delta t} + \frac{1}{\Delta x} \left( q^n_{i+1/2} - q^n_{i-1/2} \right) &= 0 \\
\frac{q^{n+1}_i - q^n_i}{\Delta t} + \frac{1}{\Delta x} \left( (R^1 f^n_{NSW})_{i+1/2} - (R^1 f^n_{NSW})_{i-1/2} \right) &= \left( \tilde{s}_0 \right)_i \delta_t q^n_1 + \left( \tilde{s}_\ell \right)_i \delta_t q^n_N
\end{align*}
\]
Numerical scheme for the reformulated system

- Second order extension: MacCormack (prediction-correction)
- Advantage: no sponge layer required

Incoming solitary wave
Numerical scheme for the reformulated system

<table>
<thead>
<tr>
<th>$\Delta x$</th>
<th>Lax-Friedrichs</th>
<th>MacCormack</th>
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</thead>
<tbody>
<tr>
<td></td>
<td>$L^2$-error</td>
<td>Order</td>
</tr>
<tr>
<td>0.569662</td>
<td>0.052002</td>
<td>–</td>
</tr>
<tr>
<td>0.284831</td>
<td>0.040773</td>
<td>0.35</td>
</tr>
<tr>
<td>0.142416</td>
<td>0.024022</td>
<td>0.76</td>
</tr>
<tr>
<td>0.071208</td>
<td>0.012777</td>
<td>0.91</td>
</tr>
<tr>
<td>0.035604</td>
<td>0.006621</td>
<td>0.95</td>
</tr>
</tbody>
</table>

**Table:** Error for incoming soliton ($\zeta$ enforced)

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<td>0.014574</td>
<td>0.73</td>
</tr>
<tr>
<td>0.071208</td>
<td>0.008471</td>
<td>0.78</td>
</tr>
<tr>
<td>0.035604</td>
<td>0.004751</td>
<td>0.83</td>
</tr>
<tr>
<td>0.017802</td>
<td>0.002559</td>
<td>0.89</td>
</tr>
</tbody>
</table>

**Table:** Error for outgoing soliton ($\zeta$ enforced)
Boussinesq-Peregrine system with varying bottom

Account for varying bottom with Boussinesq-Peregrine in \((\zeta, q)\)-coordinates

\[
\begin{align*}
\partial_t \zeta + \partial_x q &= 0 \\
(1 + h_b \mathcal{T}_b) \partial_t q + \partial_x f_{\text{NSW}} &= -gh \partial_x b
\end{align*}
\]  

in \((0, \ell)\), \hspace{1cm} \text{(BP)}

under generating boundary conditions

\[
\zeta(t, 0) = g_0(t), \quad \zeta(t, \ell) = g_\ell(t),
\]

with \(h_b = H_0 - b\) (depth at rest) and

\[
\mathcal{T}_b(\cdot) = -\frac{1}{3h_b} \partial_x \left( h_b^3 \frac{(\cdot)}{h_b} \right) + \frac{(\cdot)}{2} \partial_x^2 b,
\]

\hspace{1cm} \text{(6)}
Boussinesq-Peregrine system with varying bottom

Note $R_b^0$ the inverse of $(1 + h_b T_b)$ with homogeneous Dirichlet conditions. Then

$$\partial_t q = -R_b^0 \partial_x f_{NSW} - gR_b^0 (h \partial_x b) + s_{(b,0)} \dot{q}_0 + s_{(b,\ell)} \dot{q}_\ell$$

(7)

where

$$\left\{ \begin{array}{l}
(1 + h_b T_b) s_{(b,0)} = 0 \\
\dot{s}_{(b,0)}(0) = 1, \quad s_{(b,0)}(\ell) = 0
\end{array} \right.$$

and

$$\left\{ \begin{array}{l}
(1 + h_b T_b) s_{(b,\ell)} = 0 \\
\dot{s}_{(b,\ell)}(0) = 0, \quad s_{(b,\ell)}(\ell) = 1
\end{array} \right.$$

Lemma 1 (generalization of $R_0 \partial_x$)

We can construct a nonlocal operator $R_1 b$ such that

$$R_0 b \partial_x (\cdot) = 1 \alpha (\partial_x + \phi) [h_2 b R_1 b (\cdot)] - R_0 b (\cdot) \phi f + R_0 b (\phi f_{NSW})$$

Definition 1 (Nonlocal flux and source terms)

$$f = h_2 b R_1 b (f_{NSW} h_2 b)$$

$$S = R_0 b (-g h \partial_x b)$$

$$\left\{ \begin{array}{l}
(1 + h_b T_b) s_{(b,0)} = 0 \\
\dot{s}_{(b,0)}(0) = 1, \quad s_{(b,0)}(\ell) = 0
\end{array} \right.$$

and

$$\left\{ \begin{array}{l}
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\end{array} \right..$$
Boussinesq-Peregrine system with varying bottom

Note $R_b^0$ the inverse of $(1 + h_b T_b)$ with **homogeneous Dirichlet** conditions. Then

$$
\partial_t q = -R_b^0 \partial_x f_{\text{NSW}} - gR_b^0(h \partial_x b) + s_{(b,0)} \dot{q}_0 + s_{(b,\ell)} \dot{q}_\ell
$$

where

$$
\begin{cases}
(1 + h_b T_b) s_{(b,0)} = 0 \\
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\end{cases}
$$

and

$$
\begin{cases}
(1 + h_b T_b) s_{(b,\ell)} = 0 \\
s_{(b,\ell)}(0) = 0, \quad s_{(b,\ell)}(\ell) = 1
\end{cases}
$$

**Lemma 1 (generalization of $R_b^0 \partial_x = \partial_x R_b^1$)**

We can construct a nonlocal operator $R_b^1$ such that

$$
R_b^0 \partial_x (\cdot) = \frac{1}{\alpha} (\partial_x + \phi) \left[h_b^2 R_b^1 \left(\frac{(\cdot)}{h_b^2}\right)\right] - R_b^0 (\cdot) \phi
$$

with $\alpha = 1 + \frac{1}{4} (\partial_x b)^2$ and $\phi = \frac{3}{2} \frac{\partial_x b}{h_b}$

**Definition 1 (Nonlocal flux and source terms)**

$$
\bar{f} = h_b^2 R_b^1 \left(\frac{f_{\text{NSW}}}{h_b^2}\right), \quad \bar{S} = R_b^0 (-gh \partial_x b) - \frac{\phi}{\alpha} \bar{f} + R_b^0 (\phi f_{\text{NSW}})
$$
Proposition 2 (Equivalent formulation with nonlocal flux)

Let \((\zeta, q)\) initially equal to \((\zeta^{\text{in}}, q^{\text{in}})\). The two assertions are equivalent:

1. The pair \((\zeta, q)\) satisfies \((BP)\) with generating conditions \(\zeta(\cdot, 0) = g_0\) and \(\zeta(\cdot, \ell) = g_\ell\)

2. The pair \((\zeta, q)\) satisfies

\[
\begin{dcases}
\partial_t \zeta + \partial_x q = 0, \\
\partial_t q + \frac{1}{\alpha} \partial_x f(U, x) = \Xi(U, x) + s_{(b,0)} q_0 + s_{(b,\ell)} q_\ell
\end{dcases}
\quad\text{in } (0, \ell) \tag{8}
\]

and the trace equations

\[
\begin{pmatrix}
\dot{s}_{(b,0)}'(0) \\
\dot{s}_{(b,\ell)}'(0)
\end{pmatrix}^T
\begin{pmatrix}
\dot{q}_0 \\
\dot{q}_\ell
\end{pmatrix} = \Phi\left(q_{l_0,\ell}, f_{l_0,\ell}, \partial_x R_{b l_0,\ell}^0 (\phi f_{\text{NSW}} - gh \partial_x b)\right) - \begin{pmatrix}
\dot{g}_0 \\
\dot{g}_\ell
\end{pmatrix} \tag{9}
\]

where \(\Phi : \mathbb{R}^3 \to \mathbb{R}^2\) known.
**Question:** starting from a wrong initial condition, can we recover the solution by enforcing appropriate boundary conditions?
Enforcing waves through boundary conditions

**Question:** starting from a wrong initial condition, can we recover the solution by enforcing appropriate boundary conditions?

**Setup:**
- approximate $U_{\text{ref}}$ solution in $(-\ell, 2\ell)$ with periodic conditions;
- extract $g_0(t^n) := \sigma(t^n)\xi^+|_{U_{\text{ref}}}(t^n)$ and $g_\ell(t^n) := \sigma(t^n)\xi^-|_{U_{\text{ref}}}(t^n)$;
- approximate new solution in $(0, \ell)$, initially at rest, with $g_0$, $g_\ell$ enforced at boundaries;

**Figure:** Initial condition $((2\pi H_0)^2/\lambda^2 = 0.31, a_0/H_0 = 0.25)$
Comparing different boundary conditions

![Graph comparing different boundary conditions](image)

Free surface elevation $\zeta$ [m]

- Reference
- $\zeta$ left and right
- Riemann left and right

x position ($\ell = 28.3$ [m], $\Delta x = 0.283$ [m])
**Motivation:** wave breaking with dispersive models $\rightarrow$ non physical oscillations.

$\rightarrow$ Cancel dispersive term near shock wave
Coupling Boussinesq-Peregrine and shallow water models

\[
\begin{aligned}
\partial_t \zeta_L + \partial_x q_L &= 0 \quad \text{in } (0, \ell_1) \\
\partial_t q_L + \frac{1}{\alpha} \partial_x f(U_L) &= \Xi(U_L) + \xi_{(b,0)} \dot{q}_{L|0} + \xi_{(b,\ell_1)} \dot{q}_{L|\ell_1} \\
\partial_t \zeta_R + \partial_x q_R &= 0 \quad \text{in } (\ell_1, \ell_2) \\
\partial_t q_R + \partial_x f_{NSW}(U_R) &= -gh_R \partial_x b
\end{aligned}
\]

(10)

Coupling conditions: \( R_+(U_R)_{|\ell_1} = R_+(U_L)_{|\ell_1} \), \( R_-(U_L)_{|\ell_1} = R_-(U_R)_{|\ell_1} \)
Coupling Boussinesq-Peregrine and shallow water models

\[
\begin{aligned}
\partial_t \zeta_L + \partial_x q_L &= 0 \quad \text{in } (0, \ell_1) \\
\partial_t q_L + \frac{1}{\alpha} \partial_x f(U_L) &= \zeta(U_L) + \delta_{(b,0)} \dot{q}_{L|0} + \delta_{(b,\ell_1)} \dot{q}_{L|\ell_1} \\
\partial_t \zeta_R + \partial_x q_R &= 0 \quad \text{in } (\ell_1, \ell_2) \\
\partial_t q_R + \partial_x f_{NSW}(U_R) &= -gh_R \partial_x b
\end{aligned}
\]

Coupling conditions:

\[R_+(U_R)_{|\ell_1} = R_+(U_L)_{|\ell_1}, \quad R_-(U_L)_{|\ell_1} = R_-(U_R)_{|\ell_1}\]

In practice, **overlapping** helps to reduce oscillations/reflections
Coupling Boussinesq-Peregrine and shallow water models

\[
\begin{align*}
\frac{\partial}{\partial t} \zeta_L + \frac{\partial}{\partial x} q_L &= 0 \quad \text{in } (0, \ell_1) \\
\frac{\partial}{\partial t} q_L + \frac{1}{\alpha} \frac{\partial}{\partial x} f(U_L) &= \mathcal{E}(U_L) + \dot{s}_{b,0} \dot{q}_{L|0} + \dot{s}_{b,\ell_1} \dot{q}_{L|\ell_1} \\
\frac{\partial}{\partial t} \zeta_R + \frac{\partial}{\partial x} q_R &= 0 \quad \text{in } (\ell_1, \ell_2) \\
\frac{\partial}{\partial t} q_R + \frac{\partial}{\partial x} f_{\text{NSW}}(U_R) &= -gh_R \frac{\partial}{\partial x} b
\end{align*}
\]

\( \text{(10)} \)

Coupling conditions:
\[ R_+(U_R)_{|\ell_1} = R_+(U_L)_{|\ell_1}, \quad R_-(U_L)_{|\ell_1} = R_-(U_R)_{|\ell_1} \]

In practice, **overlapping** helps to reduce oscillations/reflections

1. Approx. \( U_{R,i}^{n+1} \) with FV scheme + hydrostatic reconstruction; \( R_+(U_R)_{|\ell_1} = R_+(U_L)_{|\ell_1} \)
2. Approx. \( U_{L,i}^{n+1} \) with Lax-Friedrichs scheme + trace equations; \( R_-(U_L)_{|\ell_1+\epsilon} = R_-(U_R)_{|\ell_1+\epsilon} \)
3. Convex combination in overlapping area: \( U_{i}^{n+1} = \rho(x_i) U_{L,i}^{n+1} + (1 - \rho(x_i)) U_{R,i}^{n+1} \).
Experimental testcase: LEGI

Initial condition

Free surface elevation $\zeta$ [m]

-0.25 to 0.05

x position [m]

-25 to 5

Boussinesq-Peregrine
NSW
Bathymetry
Gauge

$t^* = t(g/a_0)^{1/2}$

Gauge 2
Gauge 3
Gauge 4
Gauge 5
Gauge 6
Num. Approx.
Approximate transparent boundary conditions

Use coupling as a sponge layer to evacuate waves.

Figure: Outgoing soliton at times $t = 0, 9.46, 14.19, \text{ and } 23.16 \, [s]$. Green domain corresponds to NSW.
Conclusion and perspectives

Over a flat bottom:
- Reformulation of Boussinesq-Abbott
- Generalized boundary conditions
- Efficient 1st and 2nd order schemes

Over a varying bottom:
- Approach extended to Boussinesq-Peregrine
- Coupling with shallow water model
- Implementation + validation (experimental data, various boundary conditions tested)

Thank you for your attention!
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- Implement scheme in UHAINA
- Extension to Boussinesq models with improved dispersion relation
- Statistics of extreme waves: impact of bathymetry
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Thank you for your attention!
If \((h, q)\) solves the shallow water system, then the **Riemann invariants** satisfy

\[
\begin{align*}
\partial_t R_+(h, q) + \lambda_+(h, q) \partial_x R_+(h, q) &= 0 \\
\partial_t R_-(h, q) + \lambda_-(h, q) \partial_x R_-(h, q) &= 0
\end{align*}
\]

with \(R_\pm = u \pm 2 \sqrt{gh}\) and \(\lambda_\pm = u \pm \sqrt{gh}\).

- For flat bathymetry, \(R_\pm\) remains constant along characteristics.
- Natural choice of outgoing data in fluvial regime (\(|u| < \sqrt{gh}\)).
- In the shallow limit \(H_0^2 / L^2 \to 0\), Boussinesq models degenerate into shallow water.
We wish to enforce more general boundary conditions

\[ \xi^+[\zeta, q](t, 0) = g_0(t), \quad \xi^-[\zeta, q](t, \ell) = g_{\ell}(t), \quad (11) \]

Assume there exists a smooth map \( \mathcal{H} : \xi^+[\zeta, q] \mapsto (\zeta, q) : \)

\[
\begin{pmatrix}
  s'_{(b,0)}(0) & s'_{(b,\ell)}(0) \\
  s'_{(b,0)}(\ell) & s'_{(b,\ell)}(\ell)
\end{pmatrix}
\begin{pmatrix}
  \dot{\mathcal{H}}_{2|0} \\
  \dot{\mathcal{H}}_{2|\ell}
\end{pmatrix}
= \Phi
\begin{pmatrix}
  \mathcal{H}_{2|0}, f_{l_0,\ell}, \partial_x R^0_{b|0,\ell} (\phi f_{NSW} - gh \partial_x b) \\
  \mathcal{H}_{1|0}, \mathcal{H}_{1|\ell}
\end{pmatrix}
\]

Noting \( X = (\dot{\xi}_{l_0}, \dot{\xi}_{\ell})^T \), the trace equations become

\[
\begin{cases}
  \dot{X} = Y \\
  D(X) \dot{Y} + M(t, X, Y)Y = \widetilde{\Phi}(t, X)
\end{cases}
\quad \text{such that} \quad D(X), M(t, X, Y) \in \mathbb{R}^{2 \times 2}
\]

Discretize this ODE with

\[
\begin{cases}
  X^{n+1} - X^n = \frac{Y^n \Delta t}{D^n} \\
  \Delta t Y^{n+1} - Y^n = \frac{M^n Y^{n+1} + \Phi^n}{\Delta t}
\end{cases}
\]
Numerical scheme

Time stepping procedure

Step 1: Approximate $f(t^n, \cdot) = h_b^2 R_b^1 \left( \frac{f_{NSW}^{n}}{h_b^2} \right)$ and $R_b^0 (\phi f_{NSW} - gh \partial_x b)(t^n, \cdot)$ respectively with

\[
(h_{b,i}^2, v_i)_{1 \leq i \leq N} \quad \text{s.t.} \quad (R_b^1)^{-1} v = (f_{NSW,i}^{n}/h_{b,i}^2)_{1 \leq i \leq N},
\]

\[
w \in \mathbb{R}^N \quad \text{s.t.} \quad (R_b^0)^{-1} w = (\phi_i f_{NSW,i}^{n} - gh_i^{n} \delta_x b_i)_{1 \leq i \leq N}.
\]

Step 2: Approximate the trace equations to get $\delta_t (\xi^-)_{1}^{n}$ and $\delta_t (\xi^+)_{N}^{n}$. Then set

\[
\begin{cases}
(\xi^-)_{1}^{n+1} = (\xi^-)_{1}^{n} + \Delta t \delta_t (\xi^-)_{1}^{n}, \\
(\xi^+)_{N}^{n+1} = (\xi^+)_{N}^{n} + \Delta t \delta_t (\xi^+)_{N}^{n},
\end{cases}
\]

\[
\begin{cases}
(\xi^+)_{1}^{n+1} = g_0(t^{n+1}), \\
(\xi^-)_{N}^{n+1} = g_\ell(t^{n+1}),
\end{cases}
\]

\[
\begin{cases}
U_{1}^{n+1} = \mathcal{H}((\xi^\pm)_{1}^{n+1}) \\
U_{N}^{n+1} = \mathcal{H}((\xi^\pm)_{N}^{n+1})
\end{cases}
\]

Step 3: For $2 \leq i \leq N$, finite volume update with Lax-Friedrichs numerical flux

\[
\begin{cases}
\frac{\xi_{i}^{n+1} - \xi_{i}^{n}}{\Delta t} + \frac{1}{\Delta x} \left( q_{i+1/2}^{n} - q_{i-1/2}^{n} \right) = 0, \\
\frac{q_{i}^{n+1} - q_{i}^{n}}{\Delta t} + \frac{1}{\alpha_i} \frac{\tilde{f}_{i+1/2}^{n} - \tilde{f}_{i-1/2}^{n}}{\Delta x} = \mathcal{G}_{i}^{n} + (s_{(b,0)})_{i} \delta_t q_{i}^{n} + (s_{(b,\ell)})_{i} \delta_t q_{N}^{n}
\end{cases}
\]
Alternative coupling

\[ \begin{array}{cccc}
(x_L, q_L) & \overline{U} & (x_R, q_R) \\
0 & \text{Boussinesq-Peregrine} & \ell_1 & \ell_2 & \ell_3 & \text{NSW}
\end{array} \]

Coupling with overlapping:

\[
\begin{aligned}
\partial_t \zeta_L + \partial_x q_L &= 0 & x \in (0, \ell_2) \\
\partial_t \zeta_R + \partial_x q_R &= 0 & x \in (\ell_1, \ell_3) \\
\partial_t q_L + \frac{1}{\alpha} \partial_x f(\overline{U}) &= \Xi(\overline{U}) + s_{(b,0)} \dot{q}_{L|0} + s_{(b,\ell_2)} \dot{q}_{L|\ell_2} & x \in (0, \ell_2) \\
\partial_t q_R + \partial_x f_{\text{NSW}}(\overline{U}) &= -g\hbar \partial_x b & x \in (\ell_1, \ell_3)
\end{aligned}
\]

with

\[
\overline{U}(t, x) = \begin{cases} 
\sigma(x) U_L + (1 - \sigma(x)) U_R & \ell_1 \leq x \leq \ell_2 \\
U_L & x < \ell_1 \\
U_R & x > \ell_2
\end{cases}
\]

and the coupling conditions

\[
\mathcal{R}_+(U_R)|_{\ell_1} = \mathcal{R}_+(U_L)|_{\ell_1}, \quad \mathcal{R}_-(U_L)|_{\ell_2} = \mathcal{R}_-(U_R)|_{\ell_2}.
\]