Boundary conditions for Boussinesq-type models in elevation-flux form

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Ile d'Aix – 4 octobre 2023

Objectives

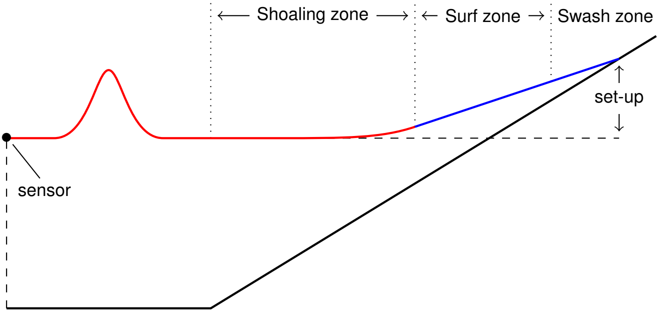
Postdoc supported by **Institut des Mathématiques de la Planète Terre**

Supervision : David Lannes and Philippe Bonneton

Long term goal: study extreme waves in littoral area

- Need accurate dispersive model: Boussinesq-type systems
- Boundary conditions are difficult to deal with

  Recently: Perfectly Matched Layer, source function method → costly



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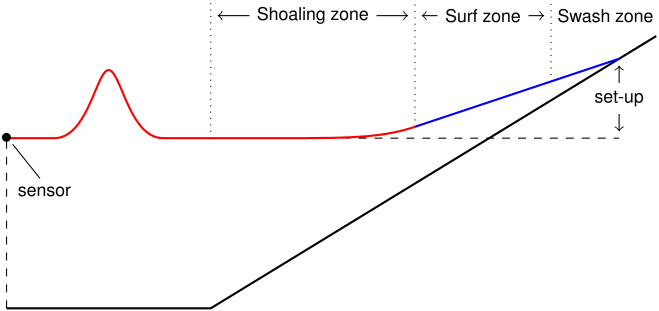
→ We propose a new and efficient method for boundary conditions.



Boussinesq-Abbott: dispersive model over flat bottom

Consider the Boussinesq-Abbott system

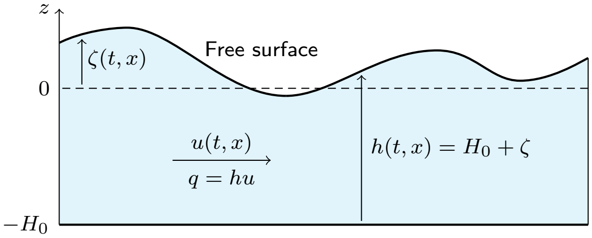
$$\begin{cases} \partial_t \zeta + \partial_x q = 0 \\ (1 - \kappa^2 \partial_{xx}^2)\partial_t q + \partial_x f_{\mathrm{NSW}}(\zeta, q) = 0 \end{cases} \quad \text{in } (0, \ell) \tag{BA}$$

with *generating boundary conditions*

$$\zeta(t, 0) = g_0(t), \qquad \zeta(t, \ell) = g_\ell(t), \tag{1}$$

where $g_0, g_\ell \in C(0, T)$ and

$$\kappa^2 = H_0^2/3, \qquad f_{\mathrm{NSW}}(\zeta, q) = hu^2 + gh^2/2$$



Reformulation of the model

How to account for boundary conditions? How to recover $q_{|x=0,\ell}$?

- Hyperbolic case ($\kappa = 0$) : Riemann invariants
- Dispersive case ($\kappa > 0$) : need to invert $(1 - \kappa^2 \partial_{xx}^2)$ $\rightarrow$ requires knowledge on $\partial_t q_{|x=0,\ell}$
  Lannes and Weynans 2020 "Generating boundary conditions for a Boussinesq system"

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Fix $0 \leq t \leq T$, then $y(x) = \partial_t q(t,x)$ satisfies an ODE of the form

$$\begin{cases} y - \kappa^2 y'' = \phi(x) \\ y(0) = \dot{q}_{|0}, \quad y(\ell) = \dot{q}_{|\ell} \end{cases}$$

Equivalently: $y = y_h + y_b$ with $\begin{cases} y_h - \kappa^2 y_h'' = 0 \\ y_h(0) = \dot{q}_{|0}, \quad y_h(\ell) = \dot{q}_{|\ell} \end{cases}$ and $\begin{cases} y_b - \kappa^2 y_b'' = \phi(x) \\ y_b(0) = y_b(\ell) = 0 \end{cases}$

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Define R^0 as the inverse of $(1 - \kappa^2 \partial_{xx}^2)$ with **homogeneous Dirichlet** conditions. Then

$$\partial_t q = \underbrace{-R^0 \partial_x f_{\mathrm{NSW}}}_{y_b} + \underbrace{\mathfrak{s}_{(0)} \dot{q}_{|0} + \mathfrak{s}_{(\ell)} \dot{q}_{|\ell}}_{y_h}$$

$$\text{where} \quad \begin{cases} (1 - \kappa^2 \partial_x^2)\mathfrak{s}_{(0)} = 0 \\ \mathfrak{s}_{(0)}(0) = 1, \quad \mathfrak{s}_{(0)}(\ell) = 0 \end{cases} \quad \text{and} \quad \begin{cases} (1 - \kappa^2 \partial_x^2)\mathfrak{s}_{(\ell)} = 0 \\ \mathfrak{s}_{(\ell)}(0) = 0, \quad \mathfrak{s}_{(\ell)}(\ell) = 1 \end{cases} . \qquad (2)$$

Reformulation of the model

Note R^1 the inverse of $(1 - \kappa^2 \partial_{xx}^2)$ with **homogeneous Neumann** conditions

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Proposition 1 (Equivalent formulation with nonlocal flux)

Let (ζ, q) initially equal to $(\zeta^{\text{in}}, q^{\text{in}})$. The two assertions are equivalent:

1. *The pair (ζ, q) satisfies (BA) with generating conditions $\zeta(\cdot, 0) = g_0$ and $\zeta(\cdot, \ell) = g_\ell$*
2. *The pair (ζ, q) satisfies*

$$\begin{cases} \partial_t \zeta + \partial_x q = 0 \\ \partial_t q + \partial_x (R^1 f_{\text{NSW}}) = \mathfrak{s}_{(0)} \dot{q}_{|_0} + \mathfrak{s}_{(\ell)} \dot{q}_{|_\ell} \end{cases} \quad \text{in } (0, \ell), \tag{3}$$

with

$$\begin{pmatrix} \mathfrak{s}'_{(0)}(0) & \mathfrak{s}'_{(\ell)}(0) \\ \mathfrak{s}'_{(0)}(\ell) & \mathfrak{s}'_{(\ell)}(\ell) \end{pmatrix} \begin{pmatrix} \dot{q}_0 \\ \dot{q}_\ell \end{pmatrix} = \frac{1}{\kappa^2} \begin{pmatrix} (R^1 - \text{id})_{|_0} f_{\text{NSW}} \\ (R^1 - \text{id})_{|_\ell} f_{\text{NSW}} \end{pmatrix} - \begin{pmatrix} \ddot{g}_0 \\ \ddot{g}_\ell \end{pmatrix} \tag{4}$$

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Sketch of the proof:

- To get (3), check that $R^0 \partial_x = \partial_x R^1$.
- Apply ∂_x to the discharge eq. from (3); take the trace at $x = 0, \ell$ to get (4).

$$\underbrace{\partial_{xt}^2 q}_{-\partial_{tt}^2 \zeta} + \underbrace{(\partial_{xx} R^1 f_{\text{NSW}})}_{\frac{1}{\kappa^2}(\text{id} - R^1) f_{\text{NSW}}} = \mathfrak{s}'_{(0)} \dot{q}_0 + \mathfrak{s}'_{(\ell)} \dot{q}_\ell$$

Other boundary conditions

Possibility to enforce general boundary conditions

$$\xi^+[\zeta, q](t, 0) = g_0(t), \qquad \xi^-[\zeta, q](t, \ell) = g_\ell(t). \tag{5}$$

For instance, $\xi^\pm$ given by q or Riemann invariants

$$\mathcal{R}_\pm(U) = u \pm 2\sqrt{gh}.$$

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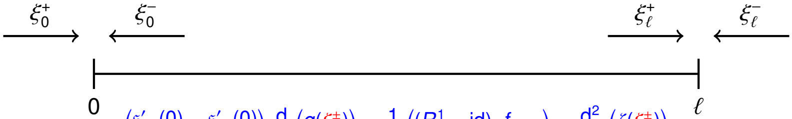
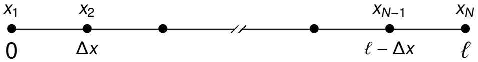
Adapt trace ODE in terms of missing data (outgoing information ξ_0^- and ξ_ℓ^+)



$$\begin{pmatrix} \mathfrak{s}'_{(0)}(0) & \mathfrak{s}'_{(\ell)}(0) \\ \mathfrak{s}'_{(0)}(\ell) & \mathfrak{s}'_{(\ell)}(\ell) \end{pmatrix} \frac{\mathrm{d}}{\mathrm{d}t} \begin{pmatrix} q(\xi_0^\pm) \\ q(\xi_\ell^\pm) \end{pmatrix} = \frac{1}{\kappa^2} \begin{pmatrix} (R^1 - \mathrm{id})_{|0} f_{\mathrm{NSW}} \\ (R^1 - \mathrm{id})_{|\ell} f_{\mathrm{NSW}} \end{pmatrix} - \frac{\mathrm{d}^2}{\mathrm{d}t^2} \begin{pmatrix} \zeta(\xi_0^\pm) \\ \zeta(\xi_\ell^\pm) \end{pmatrix}$$

Numerical scheme for the reformulated system

Discretize $(0, \ell)$ as follows:



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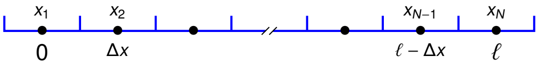

Note $U_i^n = (\zeta_i^n, q_i^n)^T$ the approximation of $\displaystyle \frac{1}{\Delta x} \int_{x_{i-1/2}}^{x_{i+1/2}} \begin{pmatrix} \zeta \\ q \end{pmatrix} (t^n, s) \, ds$.

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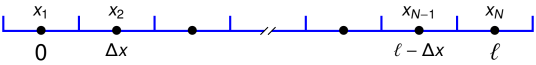

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Time stepping procedure

Step 1: Define $\underline{R}^1 f_{\mathrm{NSW}}^n$ as the vector $v \in \mathbb{R}^N$ satisfying

$$\left\{\begin{array}{l} v_i - \kappa^2 \dfrac{v_{i+1} - 2v_i + v_{i-1}}{\Delta x^2} = f_{\mathrm{NSW}}(U_i^n) \;\text{ for }\; 2 \leq i \leq N-1 \\ \dfrac{v_2 - v_1}{\Delta x} = \dfrac{v_N - v_{N-1}}{\Delta x} = 0 \end{array}\right. .$$

Similar definition for $\underline{\mathfrak{s}}_{(0)}$ and $\underline{\mathfrak{s}}_{(\ell)}$.

Numerical scheme for the reformulated system

Time stepping procedure

Step 2: Approx. trace ODEs using FD scheme to get $\delta_t q_1^n, \delta_t q_N^n$; Update border values

$$\begin{cases} q_1^{n+1} = q_1^n + \Delta t \delta_t q_1^n \\ q_N^{n+1} = q_N^n + \Delta t \delta_t q_N^n \end{cases} \quad \text{and} \quad \begin{cases} \zeta_1^{n+1} = g_0(t^{n+1}) \\ \zeta_N^{n+1} = g_\ell(t^{n+1}) \end{cases} \quad .$$

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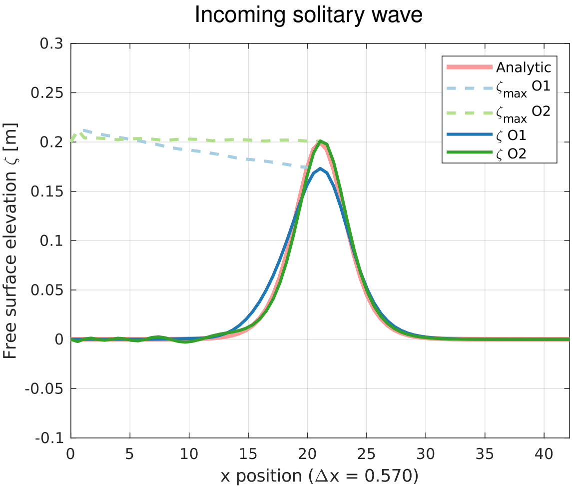
Step 3: For $2 \leq i \leq N$, finite volume update with Lax-Friedrichs numerical flux

$$\begin{cases} \dfrac{\zeta_i^{n+1} - \zeta_i^n}{\Delta t} + \dfrac{1}{\Delta x}\left(q_{i+1/2}^n - q_{i-1/2}^n\right) = 0 \\ \dfrac{q_i^{n+1} - q_i^n}{\Delta t} + \dfrac{1}{\Delta x}\left((\underline{R}^1 f_{\mathrm{NSW}}^n)_{i+1/2} - (\underline{R}^1 f_{\mathrm{NSW}}^n)_{i-1/2}\right) = (\mathfrak{s}_{(0)})_i \delta_t q_1^n + (\mathfrak{s}_{(\ell)})_i \delta_t q_N^n \end{cases} .$$

Numerical scheme for the reformulated system

- Second order extension: MacCormack (prediction-correction)
- Advantage: no sponge layer required



Numerical scheme for the reformulated system

Δx	Lax-Friedrichs		MacCormack	
	L^2-error	Order	L^2-error	Order
0.569662	0.052002	–	0.020162	–
0.284831	0.040773	0.35	0.003910	2.37
0.142416	0.024022	0.76	0.000767	2.35
0.071208	0.012777	0.91	0.000161	2.25
0.035604	0.006621	0.95	0.000037	2.12

Table: Error for incoming soliton (ζ enforced)

Δx	Lax-Friedrichs		MacCormack	
	L^2-error	Order	L^2-error	Order
0.284831	0.024246	–	0.001088	–
0.142416	0.014574	0.73	0.000314	1.79
0.071208	0.008471	0.78	0.000101	1.64
0.035604	0.004751	0.83	0.000029	1.80
0.017802	0.002559	0.89	0.000008	1.86

Table: Error for outgoing soliton (ζ enforced)

Account for varying bottom with Boussinesq-Peregrine in (ζ, q)-coordinates

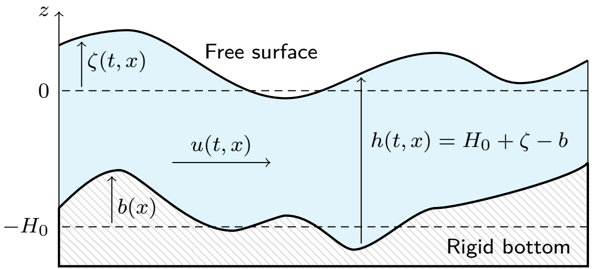
$$\begin{cases} \partial_t \zeta + \partial_x q = 0 \\ (1 + h_b \mathcal{T}_b) \partial_t q + \partial_x f_{\mathrm{NSW}} = -gh\partial_x b \end{cases} \quad \text{in } (0, \ell) \,, \tag{BP}$$

under generating boundary conditions

$$\zeta(t, 0) = g_0(t), \qquad \zeta(t, \ell) = g_\ell(t),$$

with $h_b = H_0 - b$ (depth at rest) and

$$\mathcal{T}_b(\cdot) = -\frac{1}{3h_b} \partial_x \left(h_b^3 \partial_x \frac{(\cdot)}{h_b} \right) + \frac{(\cdot)}{2} \partial_x^2 b, \tag{6}$$



Boussinesq-Peregrine system with varying bottom

Note R_b^0 the inverse of $(1 + h_b \mathcal{T}_b)$ with **homogeneous Dirichlet** conditions. Then

$$\partial_t q = -R_b^0 \partial_x f_{\mathrm{NSW}} - g R_b^0 (h \partial_x b) + \mathfrak{s}_{(b,0)} \dot{q}_{|_0} + \mathfrak{s}_{(b,\ell)} \dot{q}_{|_\ell} \tag{7}$$

$$\text{where} \quad \left\{ \begin{array}{l} (1 + h_b \mathcal{T}_b) \mathfrak{s}_{(b,0)} = 0 \\ \mathfrak{s}_{(b,0)}(0) = 1, \quad \mathfrak{s}_{(b,0)}(\ell) = 0 \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{l} (1 + h_b \mathcal{T}_b) \mathfrak{s}_{(b,\ell)} = 0 \\ \mathfrak{s}_{(b,\ell)}(0) = 0, \quad \mathfrak{s}_{(b,\ell)}(\ell) = 1 \end{array} \right. .$$

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Lemma 1 (generalization of $R^0 \partial_x = \partial_x R^1$)

We can construct a nonlocal operator R_b^1 such that

$$R_b^0 \partial_x (\cdot) = \frac{1}{\alpha} (\partial_x + \phi) \left[h_b^2 R_b^1 \left(\frac{(\cdot)}{h_b^2} \right) \right] - R_b^0 \big((\cdot) \phi \big) \quad \textit{with} \quad \alpha = 1 + \frac{1}{4} (\partial_x b)^2 \quad \textit{and} \quad \phi = \frac{3}{2} \frac{\partial_x b}{h_b}$$

Definition 1 (Nonlocal flux and source terms)

$$\mathfrak{f} = h_b^2 R_b^1 \left(\frac{f_{\mathrm{NSW}}}{h_b^2} \right), \qquad \mathfrak{S} = R_b^0 (-g h \partial_x b) - \frac{\phi}{\alpha} \mathfrak{f} + R_b^0 (\phi f_{\mathrm{NSW}})$$

Proposition 2 (Equivalent formulation with nonlocal flux)

Let (ζ, q) initially equal to $(\zeta^{\text{in}}, q^{\text{in}})$. The two assertions are equivalent:

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$$\begin{cases} \partial_t \zeta + \partial_x q = 0, \\ \partial_t q + \dfrac{1}{\alpha}\partial_x \mathfrak{f}(U, x) = \mathfrak{S}(U, x) + \mathfrak{s}_{(b,0)}\dot{q}_0 + \mathfrak{s}_{(b,\ell)}\dot{q}_\ell \end{cases} \quad \text{in } (0, \ell) \tag{8}$$

and the trace equations

$$\begin{pmatrix} \mathfrak{s}'_{(b,0)}(0) & \mathfrak{s}'_{(b,\ell)}(0) \\ \mathfrak{s}'_{(b,0)}(\ell) & \mathfrak{s}'_{(b,\ell)}(\ell) \end{pmatrix} \begin{pmatrix} \dot{q}_0 \\ \dot{q}_\ell \end{pmatrix} = \Phi\Big(q_{|0,\ell}, \mathfrak{f}_{|0,\ell}, \partial_x R^0_{b|0,\ell}(\phi f_{\text{NSW}} - gh\partial_x b)\Big) - \begin{pmatrix} \ddot{g}_0 \\ \ddot{g}_\ell \end{pmatrix} \tag{9}$$

where $\Phi : \mathbb{R}^3 \to \mathbb{R}^2$ known.

Enforcing waves through boundary conditions

Question: starting from a wrong initial condition, can we recover the solution by enforcing appropriate boundary conditions?

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Setup:

- approximate U_{ref} solution in $(-\ell, 2\ell)$ with periodic conditions;
- extract $g_0(t^n) := \sigma(t^n)\xi^+[U_{\text{ref}}]_{|_0}(t^n)$ and $g_\ell(t^n) := \sigma(t^n)\xi^-[U_{\text{ref}}]_{|_\ell}(t^n)$;
- approximate new solution in $(0, \ell)$, initially at rest, with g_0, g_ℓ enforced at boundaries;

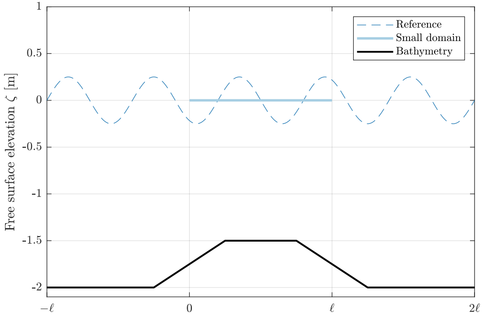
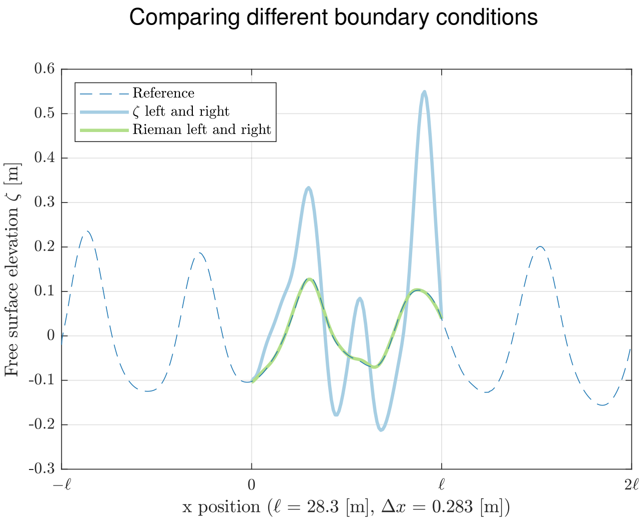
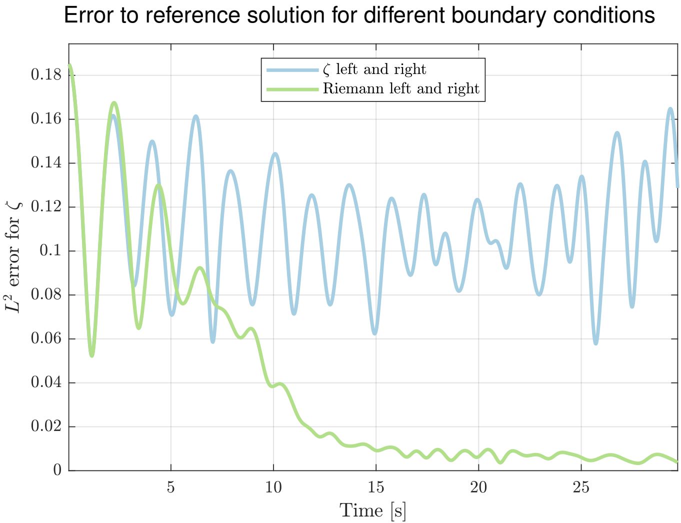


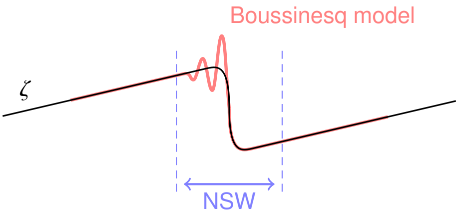
Figure: Initial condition ($(2\pi H_0)^2/\lambda^2 = 0.31$, $a_0/H_0 = 0.25$)


Comparing different boundary conditions
Reference
ζ left and right
Rieman left and right
0.6
0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
Free surface elevation ζ [m]
−ℓ
0
ℓ
2ℓ
x position (ℓ = 28.3 [m], Δx = 0.283 [m])

Enforcing waves through boundary conditions



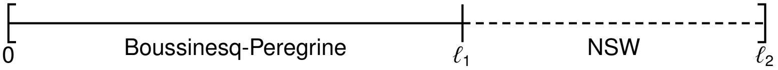
Motivation: wave breaking with dispersive models → non physical oscillations.



→ Cancel dispersive term near shock wave

$$\begin{cases} \partial_t \zeta_L + \partial_x q_L = 0 & \\ \partial_t q_L + \frac{1}{\alpha}\partial_x \mathfrak{f}(U_L) = \mathfrak{S}(U_L) + \mathfrak{s}_{(b,0)}\dot{q}_{L|_0} + \mathfrak{s}_{(b,\ell_1)}\dot{q}_{L|_{\ell_1}} & \text{in } (0,\ell_1) \\ \partial_t \zeta_R + \partial_x q_R = 0 & \\ \partial_t q_R + \partial_x f_{NSW}(U_R) = -g h_R \partial_x b & \text{in } (\ell_1,\ell_2) \end{cases} \tag{10}$$

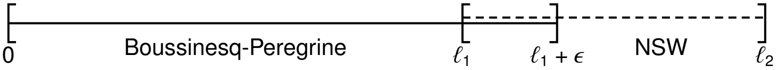
Coupling conditions: $\mathcal{R}_+(U_R)_{|_{\ell_1}} = \mathcal{R}_+(U_L)_{|_{\ell_1}}, \quad \mathcal{R}_-(U_L)_{|_{\ell_1}} = \mathcal{R}_-(U_R)_{|_{\ell_1}}$



Coupling Boussinesq-Peregrine and shallow water models

$$
\begin{cases}
\partial_t \zeta_L + \partial_x q_L = 0 \\
\partial_t q_L + \frac{1}{\alpha}\partial_x \mathfrak{f}(U_L) = \mathfrak{S}(U_L) + \mathfrak{s}_{(b,0)} \dot{q}_{L|_0} + \mathfrak{s}_{(b,\ell_1)} \dot{q}_{L|_{\ell_1}} & \text{in } (0,\ell_1) \\
\partial_t \zeta_R + \partial_x q_R = 0 \\
\partial_t q_R + \partial_x f_{NSW}(U_R) = -g h_R \partial_x b & \text{in } (\ell_1,\ell_2)
\end{cases}
\tag{10}
$$

Coupling conditions: $\mathcal{R}_+(U_R)_{|_{\ell_1}} = \mathcal{R}_+(U_L)_{|_{\ell_1}}, \quad \mathcal{R}_-(U_L)_{|_{\ell_1}} = \mathcal{R}_-(U_R)_{|_{\ell_1}}$



In practice, **overlapping** helps to reduce oscillations/reflections

$$\begin{cases} \partial_t \zeta_L + \partial_x q_L = 0 & \\ \partial_t q_L + \frac{1}{\alpha}\partial_x \mathfrak{f}(U_L) = \mathfrak{S}(U_L) + \mathfrak{s}_{(b,0)}\dot{q}_{L|0} + \mathfrak{s}_{(b,\ell_1)}\dot{q}_{L|\ell_1} & \text{in } (0,\ell_1) \\ \partial_t \zeta_R + \partial_x q_R = 0 & \\ \partial_t q_R + \partial_x f_{NSW}(U_R) = -g h_R \partial_x b & \text{in } (\ell_1,\ell_2) \end{cases} \tag{10}$$

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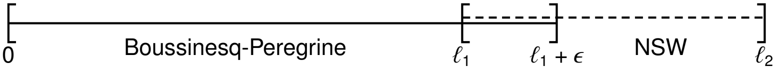
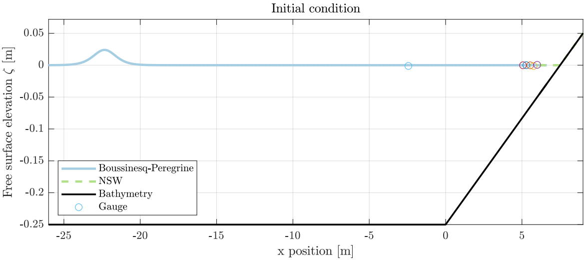
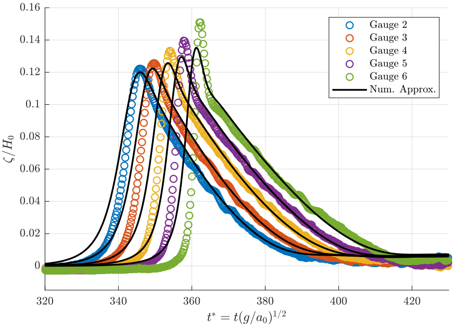


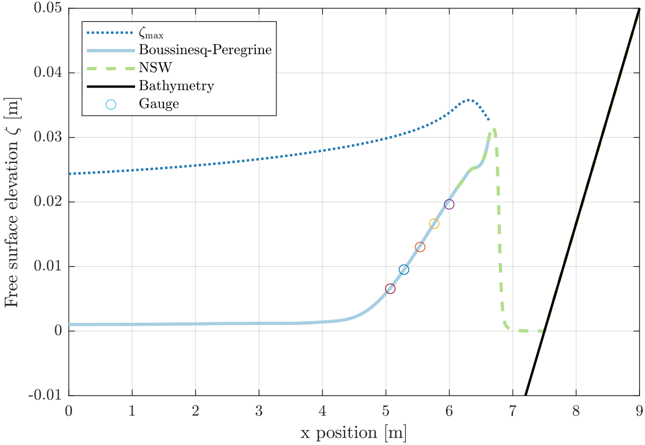
In practice, **overlapping** helps to reduce oscillations/reflections

1. Approx. U_R^{n+1} with FV scheme + hydrostatic reconstruction; $\mathcal{R}_+(U_R)_{|\ell_1} = \mathcal{R}_+(U_L)_{|\ell_1}$
2. Approx. U_L^{n+1} with Lax-Friedrichs scheme + trace equations; $\mathcal{R}_-(U_L)_{|\ell_1+\epsilon} = \mathcal{R}_-(U_R)_{|\ell_1+\epsilon}$
3. Convex combination in overlapping area: $U_i^{n+1} = \rho(x_i)U_{L,i}^{n+1} + (1-\rho(x_i))U_{R,i}^{n+1}$.


Initial condition
Free surface elevation ζ [m]
x position [m]
Boussinesq-Peregrine
NSW
Bathymetry
Gauge
0.05
0
-0.05
-0.1
-0.15
-0.2
-0.25
-25
-20
-15
-10
-5
0
5


ζ/H0
t* = t(g/a0)^1/2
Gauge 2
Gauge 3
Gauge 4
Gauge 5
Gauge 6
Num. Approx.
0.16
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
320
340
360
380
400
420

Experimental testcase: LEGI



Approximate transparent boundary conditions

Use coupling as a sponge layer to evacuate waves.

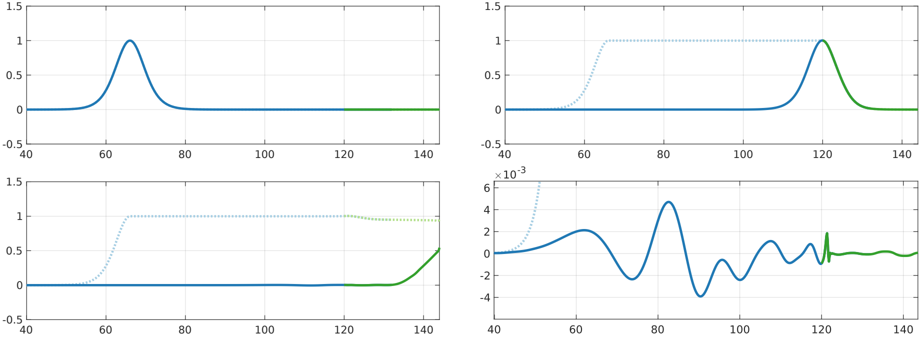


Figure: Outgoing soliton at times t = 0, 9.46, 14.19, and 23.16 [s]. Green domain corresponds to NSW.

Conclusion and perspectives

Over a flat bottom:

- Reformulation of Boussinesq-Abbott
- Generalized boundary conditions
- Efficient 1st and 2nd order schemes

Over a varying bottom:

- Approach extended to Boussinesq-Peregrine
- Coupling with shallow water model
- Implementation + validation (experimental data, various boundary conditions tested)

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Thank you for your attention!

Riemann invariants

If (h, q) solves the shallow water system, then the Riemann invariants satisfy

$$\begin{cases} \partial_t \mathcal{R}_+(h,q) + \lambda_+(h,q)\partial_x \mathcal{R}_+(h,q) = 0 \\ \partial_t \mathcal{R}_-(h,q) + \lambda_-(h,q)\partial_x \mathcal{R}_-(h,q) = 0 \end{cases}$$

with $\mathcal{R}_\pm = u \pm 2\sqrt{gh}$ and $\lambda_\pm = u \pm \sqrt{gh}$.

- For flat bathymetry, $\mathcal{R}_\pm$ remains constant along characteristics.
- Natural choice of outgoing data in fluvial regime ($|u| < \sqrt{gh}$).
- In the shallow limit $H_0^2/L^2 \to 0$, Boussinesq models degenerate into shallow water.

General boundary conditions

We wish to enforce more general boundary conditions

$$\xi^+[\zeta,q](t,0) = g_0(t), \qquad \xi^-[\zeta,q](t,\ell) = g_\ell(t), \tag{11}$$

Assume there exists a smooth map $\mathcal{H} : \xi^\pm[\zeta,q] \mapsto (\zeta,q)$:

$$\begin{pmatrix} \mathfrak{s}'_{(b,0)}(0) & \mathfrak{s}'_{(b,\ell)}(0) \\ \mathfrak{s}'_{(b,0)}(\ell) & \mathfrak{s}'_{(b,\ell)}(\ell) \end{pmatrix} \begin{pmatrix} \dot{\mathcal{H}}_{2|_0} \\ \dot{\mathcal{H}}_{2|_\ell} \end{pmatrix} = \Phi\Big(\mathcal{H}_{2|_{0,\ell}}, \mathfrak{f}_{|_{0,\ell}}, \partial_x R^0_{b|_{0,\ell}}(\phi f_{\mathrm{NSW}} - gh\partial_x b)\Big) - \begin{pmatrix} \ddot{\mathcal{H}}_{1|_0} \\ \ddot{\mathcal{H}}_{1|_\ell} \end{pmatrix}$$

Noting $X = (\dot{\xi}^-_{|_0}, \dot{\xi}^+_{|_\ell})^T$, the trace equations become

$$\begin{cases} \dot{X} = Y \\ \mathbf{D}(X)\dot{Y} + \mathbf{M}(t,X,Y)Y = \widetilde{\Phi}(t,X) \end{cases}, \qquad \mathbf{D}(X), \mathbf{M}(t,X,Y) \in \mathbb{R}^{2\times 2}$$

Discretize this ODE with

$$\begin{cases} \dfrac{X^{n+1} - X^n}{\Delta t} = Y^n \\ \mathbf{D}^n \dfrac{Y^{n+1} - Y^n}{\Delta t} + \mathbf{M}^n Y^{n+1} = \widetilde{\Phi}^n \end{cases},$$

Numerical scheme

Time stepping procedure

Step 1: Approximate $\mathfrak{f}(t^n,\cdot) = h_b^2 R_b^1\left(\frac{f_{\mathrm{NSW}}}{h_b^2}\right)$ and $R_b^0(\phi f_{\mathrm{NSW}} - gh\partial_x b)(t^n,\cdot)$ respectively with

$$\begin{aligned} (h_{b,i}^2 v_i)_{1\le i\le N} \quad &\text{s.t.} \quad (\underline{R}_b^1)^{-1} v = (f_{\mathrm{NSW},i}^n / h_{b,i}^2)_{1\le i\le N}, \\ w\in\mathbb{R}^N \quad &\text{s.t.} \quad (\underline{R}_b^0)^{-1} w = (\phi_i f_{\mathrm{NSW},i}^n - g h_i^n \delta_x b_i)_{1\le i\le N}. \end{aligned}$$

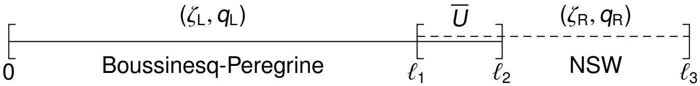
Step 2: Approximate the trace equations to get $\delta_t(\xi^-)_1^n$ and $\delta_t(\xi^+)_N^n$. Then set

$$\begin{cases} (\xi^-)_1^{n+1} = (\xi^-)_1^n + \Delta t \delta_t(\xi^-)_1^n \\ (\xi^+)_N^{n+1} = (\xi^+)_N^n + \Delta t \delta_t(\xi^+)_N^n \end{cases}, \quad \begin{cases} (\xi^+)_1^{n+1} = g_0(t^{n+1}) \\ (\xi^-)_N^{n+1} = g_\ell(t^{n+1}) \end{cases}, \quad \begin{cases} U_1^{n+1} = \mathcal{H}((\xi^\pm)_1^{n+1}) \\ U_N^{n+1} = \mathcal{H}((\xi^\pm)_N^{n+1}) \end{cases}.$$

Step 3: For $2 \le i \le N$, finite volume update with Lax-Friedrichs numerical flux

$$\begin{cases} \dfrac{\zeta_i^{n+1} - \zeta_i^n}{\Delta t} + \dfrac{1}{\Delta x}\left(q_{i+1/2}^n - q_{i-1/2}^n\right) = 0 \\ \dfrac{q_i^{n+1} - q_i^n}{\Delta t} + \dfrac{1}{\alpha_i}\dfrac{\mathfrak{f}_{i+1/2}^n - \mathfrak{f}_{i-1/2}^n}{\Delta x} = \mathfrak{S}_i^n + (\mathfrak{s}_{(b,0)})_i \delta_t q_1^n + (\mathfrak{s}_{(b,\ell)})_i \delta_t q_N^n \end{cases}.$$

Alternative coupling



Coupling with overlapping:

$$\begin{cases} \partial_t \zeta_L + \partial_x q_L = 0 & x \in (0, \ell_2) \\ \partial_t \zeta_R + \partial_x q_R = 0 & x \in (\ell_1, \ell_3) \\ \partial_t q_L + \frac{1}{\alpha}\partial_x \mathfrak{f}(\overline{U}) = \mathfrak{S}(\overline{U}) + \mathfrak{s}_{(b,0)} \dot{q}_{L|_0} + \mathfrak{s}_{(b,\ell_2)} \dot{q}_{L|_{\ell_2}} & x \in (0, \ell_2) \\ \partial_t q_R + \partial_x f_{NSW}(\overline{U}) = -g\overline{h}\partial_x b & x \in (\ell_1, \ell_3) \end{cases}$$

with

$$\overline{U}(t,x) = \begin{cases} \sigma(x) U_L + (1-\sigma(x)) U_R & \ell_1 \leq x \leq \ell_2 \\ U_L & x < \ell_1 \\ U_R & x > \ell_2 \end{cases}$$

and the coupling conditions

$$\mathcal{R}_+(U_R)_{|_{\ell_1}} = \mathcal{R}_+(U_L)_{|_{\ell_1}}, \quad \mathcal{R}_-(U_L)_{|_{\ell_2}} = \mathcal{R}_-(U_R)_{|_{\ell_2}}.$$